# EE5330 Oct. 28, 2013

## Example

$x_c(t) = \cos 2\pi F_0 t$

$x_c(nT) = \cos 2\pi F_0 nT$

$= \cos 2\pi \frac{F_0}{F_s} n$

$= \cos 2\pi f_0 n$ where $f_0 = \frac{F_0}{F_s}$

$= \cos \omega_0 n$

Recall the transform pair $\cos 2\pi F_0 t \longleftrightarrow \frac{1}{2}\left[\delta(F-F_0) + \delta(F+F_0)\right]$

We need to derive $X(e^{j\omega})$ starting from the above $X_c(F)$.

$$X_s(F) = \frac{1}{2T}\left[\delta(F-F_0) + \delta(F+F_0)\right] \qquad -\frac{F_s}{2} \leq F \leq \frac{F_s}{2}$$

Now consider $\left.\frac{1}{2}\delta(F-F_0)\right|_{F \to \frac{\omega}{2\pi T}} = \frac{1}{2T}\delta\left(\frac{\omega}{2\pi T} - F_0\right) = \pi\,\delta\left(\omega - 2\pi\frac{F_0}{F_s}\right)$

$$= \pi\,\delta(\omega - \omega_0)$$

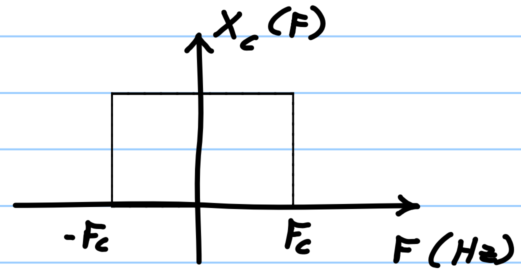
since $\delta(ax+b) = \frac{1}{|a|}\delta\left(x + \frac{b}{a}\right)$ and $\omega_0 = 2\pi f_0 = 2\pi\frac{F_0}{F_s}$

Similarly, $\frac{1}{2T}\delta(F_0 - F_s) \longrightarrow \pi\,\delta(\omega - \omega_0)$.

Thus, $\cos\omega_0 n \longleftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$, as expected.

## Example

$$x_c(t) = \frac{\sin 2\pi F_c t}{\pi t}$$

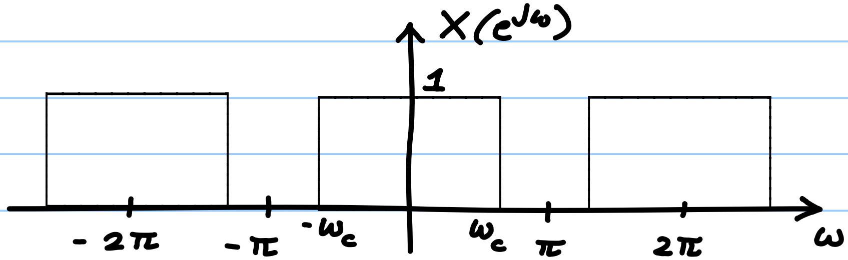



$$X_s(F) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(F - kF_s)$$

$$\frac{1}{T} \frac{\sin 2\pi \frac{F_c}{F_s} n}{\pi n} \longleftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} rect(F - kF_s)$$

Hence

$$\frac{\sin \omega_c n}{\pi n} \longleftrightarrow$$



## Example

$$x_c(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow \text{Sinc}(F) = \frac{\sin \pi F}{\pi F}$$

Since the signal is not bandlimited, sampling will cause aliasing.

We need to derive the pair $x[n] = \begin{cases} 1 & -N \leq n \leq N \\ 0 \end{cases} \longleftrightarrow \dfrac{\sin (2N+1)\omega/2}{\sin \omega/2}$

starting from the given $x_c(t)$.

$$X_c(F) = \frac{\sin \pi F}{\pi F}$$

$$X_s(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\sin \pi (F - kF_s)}{\pi (F - kF_s)}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \frac{\sin \pi\left(\frac{\omega}{2\pi T} - \frac{k}{T}\right)}{\pi\left(\frac{\omega}{2\pi T} - \frac{k}{T}\right)}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\omega - 2\pi k}{2T}\right)}{\left(\frac{\omega - 2\pi k}{2}\right)}$$

Thus,

$$\boxed{\frac{\sin (2N+1)\omega/2}{\sin \omega/2} = \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\omega - 2\pi k}{2T}\right)}{\left(\frac{\omega - 2\pi k}{2}\right)}}$$

Thus, the Dirichlet kernel is the periodic function formed from the analog sinc function.

In the above example we have assumed that none of the sampling points fall on a discontinuity.

## Sampling at a discontinuity

Recall that

$$f(t) = \text{P.V.} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\Omega) e^{j\Omega t} dt$$

where P.V. stands for "Principal Value." The LHS is not really $f(t)$, but

$$\frac{f(t^+) + f(t^-)}{2}$$

This equals $f(t)$ only if $f(t)$ is $\boxed{\text{continuous}}$ at $t$. Otherwise, the inverse transform yields the average of the function values on either side of the discontinuity.

Recall that the sampled signal spectrum is periodic. It can therefore be expressed as a Fourier series. The coefficients of the Fourier series are not arbitrary but closely related to the time function.

Recall the following Fourier series:

$$\sum_{n=-\infty}^{\infty} \delta(\Omega + n\Omega_1) = \frac{1}{\Omega_1} \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1} \quad \text{where} \quad T_1 = \frac{2\pi}{\Omega_1}$$

Hence,

$$F(\Omega) * \sum_{n=-\infty}^{\infty} \delta(\Omega + n\Omega_1) = \frac{1}{\Omega_1} F(\Omega) * \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1}$$

$$\sum_{n=-\infty}^{\infty} F(\Omega + n\Omega_1) = \sum_{k=-\infty}^{\infty} F(\Omega) * e^{-jk\Omega T_1} \cdot \frac{1}{\Omega_1}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{\Omega_1} \int_{-\infty}^{\infty} e^{-jkT_1(\Omega - y)} F(y)\,dy$$

$$= \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1} \frac{1}{\Omega_1} \int_{-\infty}^{\infty} F(y) e^{jkT_1 y}\,dy$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{\Omega_1} f(kT_1) e^{-jk\Omega T_1}$$

We have to replace the above sample value $f(kT_1)$ with its average value if $kT_1$ falls on a discontinuity.

Thus if $x_c(t) = e^{-at}u(t) \longleftrightarrow \dfrac{1}{a+j\Omega}$ and $X(e^{j\omega})$ is

is obtained as $\dfrac{1}{T}\sum_{k=-\infty}^{\infty} \dfrac{1}{a+j2\pi(F-kF_s)}$, the spectrum corresponds to a sequence whose sample value at $n=0$ is $\frac{1}{2}$, and not $1$.