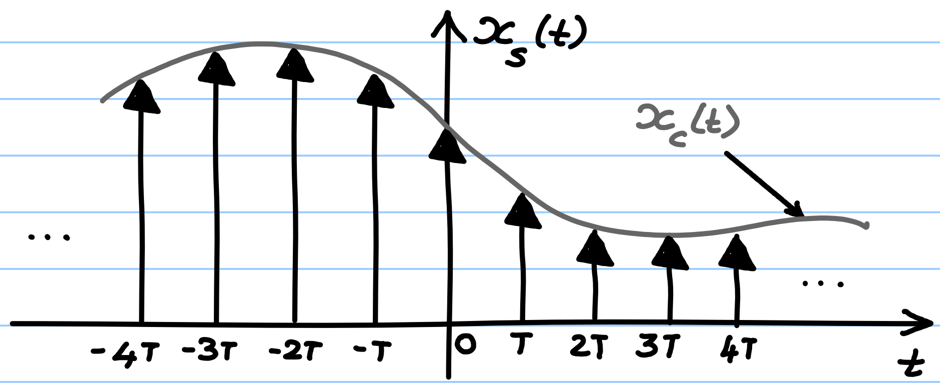
Sampling:

The process of sampling provides the bridge between the CT and DT domains. To connect the spectrum of a CT signal with that of the DT sequence's spectrum, we use the theoretical framework of impulse-train sampling.

F, Ω : CTFT frequency $\Omega = 2\pi F$

f, ω : DTFT frequency $\omega = 2\pi f$



$$x_c(t) \longleftrightarrow X_c(F)$$

$$x_s(t) = x_c(t) \cdot p(t)$$

where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$\Rightarrow \quad x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t - nT)$$

$$X_s(F) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j 2\pi F t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j 2\pi \left(\frac{F}{F_s}\right) n} \quad \text{since } F_s = \frac{1}{T}$$

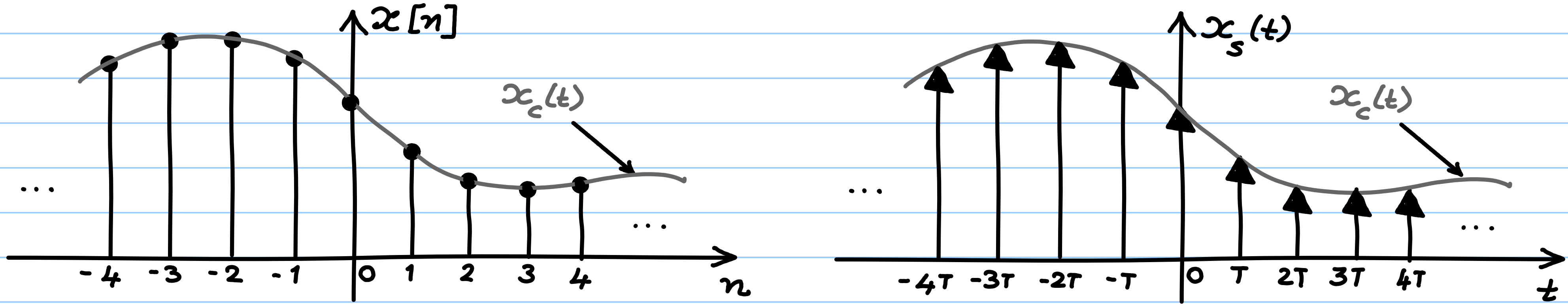
Alternatively, $p(t) \longleftrightarrow P(F) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(F - nF_s)$

Hence $x_c(t) \cdot p(t) \longleftrightarrow X_c(F) * P(F)$

Thus $$\boxed{X_s(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(F - kF_s) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j 2\pi \left(\frac{F}{F_s}\right) n}}$$

How does the spectrum of the impulse-train sampled signal relate to the spectrum of a sequence whose values are $x[n] = x_c(nT)$?

That is, are $X(e^{j\omega})$ and $X_s(F)$ related ?



To relate the spectra of the above signals, define $x[n] \triangleq x_c(nT)$. Hence,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

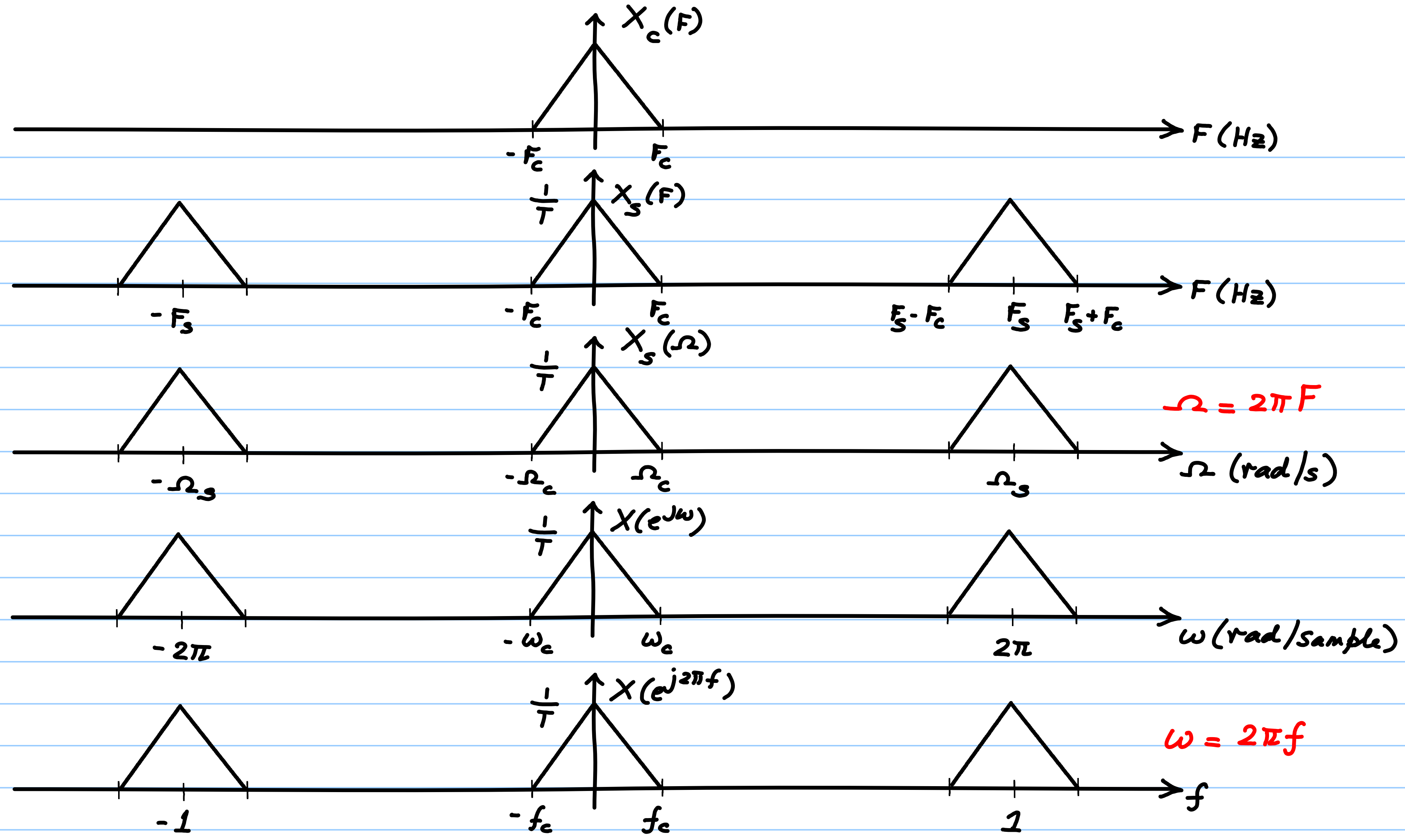
Recall $X_s(F) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j2\pi\left(\frac{F}{F_s}\right)n}$

Clearly, then,

$$X(e^{j\omega}) = X_s(F)\Big|_{F \to \frac{\omega}{2\pi T}}$$

Since $f = \frac{\omega}{2\pi}$ and $F_s = \frac{1}{T}$, the above change of variable converts the F_s-periodic $X_s(F)$ into the 2π-periodic $X(e^{j\omega})$.

If we plot the DTFT as a function $f = \frac{\omega}{2\pi}$, the period becomes 1. These are summarized in the figure below:


$X_c(F)$
F (Hz)
$-F_c$
F_c
$\frac{1}{T}$
$X_s(F)$
F (Hz)
$-F_s$
$-F_c$
F_c
$F_s - F_c$
F_s
$F_s + F_c$
$\frac{1}{T}$
$X_s(\Omega)$
$\Omega = 2\pi F$
Ω (rad/s)
$-\Omega_s$
$-\Omega_c$
Ω_c
Ω_s
$\frac{1}{T}$
$X(e^{j\omega})$
ω (rad/sample)
-2π
$-\omega_c$
ω_c
2π
$\frac{1}{T}$
$X(e^{j2\pi f})$
$\omega = 2\pi f$
f
-1
$-f_c$
f_c
1

The change of variable $F \longrightarrow \frac{\omega}{2\pi T}$ means that $X(e^{j\omega})\Big|_{\omega = 2\pi} = X_s(F_s)$.

That is, $X(e^{j\omega})$ is obtained from $X_s(F)$ by scaling it by F_s.

Thus, an analog frequency F_0 Hz gets mapped to $\boxed{\omega_0 = 2\pi \frac{F_0}{F_s}}$

Note that ω_0 is a dimensionless quantity.

The same analog frequency F_0 Hz gets mapped to a different frequency if F_s changes. In particular, if $F_{s_2} > F_{s_1}$, then

$$2\pi \frac{F_0}{F_{s_2}} < 2\pi \frac{F_0}{F_{s_1}}$$

Thus, a bandlimited spectrum with BW F_c gets mapped to a bandlimited spectrum with BW $\frac{F_c}{F_{s_1}} \cdot 2\pi$ [ω notation]. However, the same analog spectrum gets converted to a spectrum with narrower bandwidth $\frac{F_c}{F_{s_2}} \cdot 2\pi$ if sampled at $F_{s_2} > F_{s_1}$.

Thus, excessively high sampling frequencies leads to excessively narrowband spectra, making processing more difficult (in terms of filters needed, processing speed, etc.)