EE5330 Oct. 22, 2013

Note Title 22-10-2013

Consider the following causal and stable all-pass filter:

$$x[n] \longrightarrow \boxed{h_{ap}[n]} \longrightarrow y[n]$$

We showed earlier that $\displaystyle\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2$

Since the filter is causal, and the i/p is applied at $n=0$, the lower limit can be replaced by $n=0$.

Now consider the following i/p: $x_1[n] = \begin{cases} x[n] & n \le n_0 \\ 0 & n > n_0 \end{cases}$

Let $y_1[n]$ be the corresponding output. Then,

$$\sum_{n=0}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} |y_1[n]|^2$$

$$\sum_{n=0}^{n_0} |x[n]|^2 = \sum_{n=0}^{n_0} |y_1[n]|^2 + \sum_{n=n_0+1}^{\infty} |y_1[n]|^2$$

$$= \sum_{n=0}^{n_0} |y[n]|^2 + \sum_{n=n_0+1}^{\infty} |y_1[n]|^2$$

[Since $y_1[n] = y[n]$ for $n \leq n_0$]

$$\geq \sum_{n=0}^{n_0} |y[n]|^2$$

Thus, for an all-pass filter, $\boxed{\sum_{n=0}^{n_0} |x[n]|^2 \geq \sum_{n=0}^{n_0} |y[n]|^2}$

The above result can be used to show that min^m phase filters have the least energy delay.

Recall that any $H(z)$ can be decomposed as follows:

$$x[n] \longrightarrow \boxed{H_{min}(z)} \xrightarrow{x_1[n]} \boxed{H_{ap}(z)} \longrightarrow y[n]$$

$y[n]$ is the o/p of an arbitrary, causal, stable filter

$x_1[n]$ is the o/p of the minimum phase counterpart of $H(z)$.

Using the previous result,

$$\sum_{n=0}^{n_0} |x_1[n]|^2 \geq \sum_{n=0}^{n_0} |y[n]|^2$$

That is, the minimum-phase filter has the least energy lag.

Hence the term "minimum lag" is more accurate than the well-entrenched "minimum phase" terminology.

"Causal" DTFT and its implications

Recall that $x[n]=0$ for $n<0$ imposed restrictions on the corresponding transform's real and imaginary parts.

Suppose now that $X(e^{j\omega})=0$ for $\omega<0$, i.e., $X(e^{j\omega})$ is "causal".

Since $X(e^{j\omega})$ is periodic, "causal" here means $X(e^{j\omega})=0$ for $-\pi<\omega<0$. Similar to expressing $x[n]$ as $x_e[n]+x_o[n]$, consider

$$X_e(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$$

$$X_o(e^{j\omega}) = \frac{1}{2j}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$$

We can recover $X(e^{j\omega})$ over $0<\omega<\pi$ from either $X_e(e^{j\omega})$ or $X_o(e^{j\omega})$:

$$X(e^{j\omega}) = \begin{cases} 2X_e(e^{j\omega}) & 0<\omega<\pi \\ 0 & -\pi<\omega<0 \end{cases}$$

$$X(e^{j\omega}) = \begin{cases} 2j\,X_o(e^{j\omega}) & 0<\omega<\pi \\ 0 & -\pi<\omega<0 \end{cases}$$

One can also relate $X_e(e^{j\omega})$ and $X_o(e^{j\omega})$.

It is easy to see that

$$X_o(e^{j\omega}) = \begin{cases} -j X_e(e^{j\omega}) & 0 < \omega < \pi \\ j X_e(e^{j\omega}) & -\pi < \omega < 0 \end{cases}$$

That is,

$$X_o(e^{j\omega}) = X_e(e^{j\omega}) H(e^{j\omega})$$

where

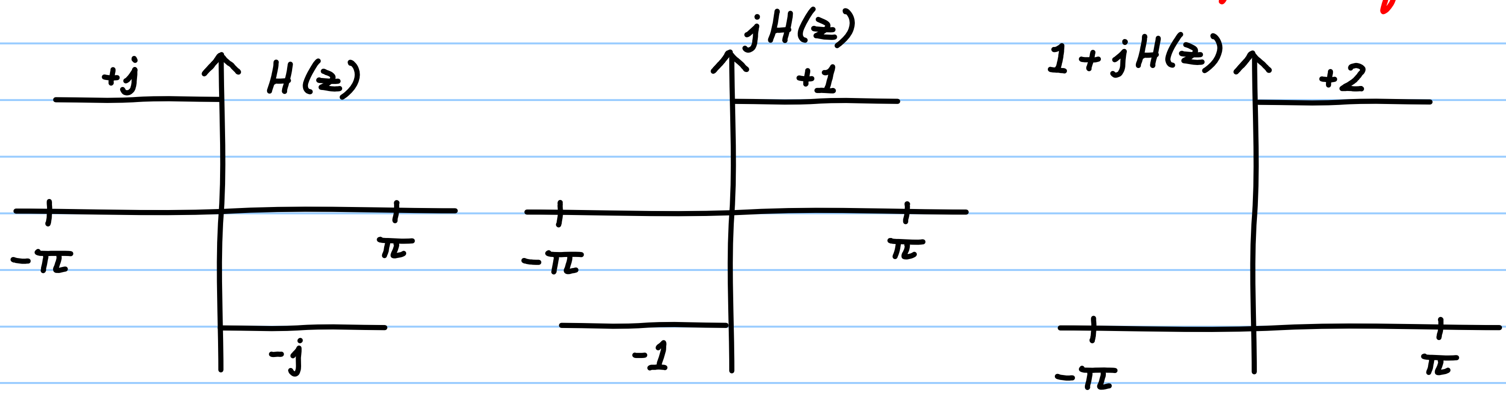
$$H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

Note that $x[n] = x_R[n] + j\, x_I[n]$

$$x_R[n] \longleftrightarrow X_e(e^{j\omega})$$

$$x_I[n] \longleftrightarrow X_o(e^{j\omega})$$

Complex Half Band Filter



Let $G(z) = 1 + jH(z)$, whence it follows $G(e^{j\omega}) = \begin{cases} 2 & 0 < \omega < \pi \\ 0 & -\pi < \omega < 0 \end{cases}$

Hence if an arbitrary $x[n]$ is filtered using $G(e^{j\omega})$, the output signal's DTFT becomes "causal" (or "one-sided").

$$x[n] \longrightarrow \boxed{1 + jH(z)} \longrightarrow y[n]$$

$$g[n] = \delta[n] + jh[n]$$

Hence,

$$y[n] = x[n] * g[n]$$

$$y[n] = \left(\delta[n] + j\,h[n]\right) * x[n]$$

$$= x[n] + j\,x[n] * h[n]$$

$$= x[n] + j\,\hat{x}[n] = x_R[n] + j\,x_I[n] \Rightarrow \boxed{x_R[n] \;\&\; x_I[n] \text{ are not independent}}$$

where $\hat{x}[n] = x[n] * h[n]$

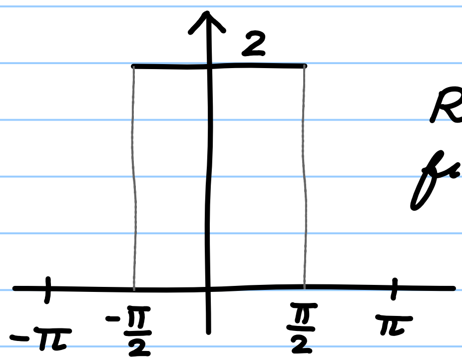
Since $H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$, one can easily verify that

$$h[n] = \begin{cases} \dfrac{\sin^2(n\pi/2)}{n\pi/2} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

$\boxed{h[n] \leftrightarrow H(e^{j\omega}) \text{ is called as the IDEAL HILBERT TRANSFORMER}}$

Exercise

Explore the relationship between the real-valued halfband filter, complex halfband filter, and the hilbert transformer. The response of a real halfband filter is given below.



Real half-band filter