EE5330 Oct. 15, 2013

Note Title 15-10-2013

Causal, stable, linear phase filters with rational transfer functions have to be necessarily FIR.

We will study more about all-pass and min^m phase filters.

A K^{th} order all-pass filter can be written as a cascade of K first order all-pass sections.

$$H_k(z) = \frac{-a_k^* + z^{-1}}{1 - a_k z^{-1}} = \frac{z^{-1} - r_k e^{-j\theta_k}}{1 - r_k e^{j\theta_k} z^{-1}}$$

$$H_{ap}(z) = \prod_{k=1}^{K} H_k(z)$$

$$\angle H_k(e^{j\omega}) = \phi_k(\omega) = \arg\{e^{-j\omega} - r_k e^{-j\theta_k}\} - \arg\{1 - r_k e^{j\theta_k} e^{-j\omega}\}$$

$$= \arg\{e^{-j\omega}\} + \arg\{1 - r_k e^{-j\theta_k} e^{j\omega}\} - \arg\{1 - r_k e^{j\theta_k} e^{-j\omega}\}$$

$$= -\omega - 2\tan^{-1} \frac{r_k \sin(\omega - \theta_k)}{1 - r_k \cos(\omega - \theta_k)}$$

The overall phase response is,

$$\phi(\omega) = -K\omega - 2\sum_{k=1}^{K} \tan^{-1} \frac{r_k \sin(\omega - \theta_k)}{1 - r_k \cos(\omega - \theta_k)}$$

The associated group delay is

$$\tau_g(\omega) = -\frac{d}{d\omega} \phi(\omega)$$

$$= K + 2\sum_{k=1}^{K} \frac{r_k \cos(\omega-\theta_k) - r_k^2}{1 - 2r_k \cos(\omega-\theta_k) + r_k^2}$$

$$= \sum_{k=1}^{K} \frac{1 - r_k^2}{1 - 2r_k \cos(\omega-\theta_k) + r_k^2} = \sum_{k=1}^{K} \frac{1 - r_k^2}{\left|1 - r_k e^{j\theta_k} e^{-j\omega}\right|^2}$$

Since $r_k < 1 \ \forall k$, $\tau_g(\omega) > 0$ for an all-pass filter

Also, since $\tau_g(\omega) = -\phi'(\omega)$, $\phi(\omega)$ is a monotonic decreasing function.

One can also easily prove the following:

$$|H_k(z)| = \begin{cases} > 1 & |z| < 1 \\ = 1 & |z| = 1 \\ < 1 & |z| > 1 \end{cases} \Rightarrow \text{this property holds for } \underbrace{H_{ap}(z)}_{K^{th}\text{ order all-pass}} \text{ also}$$

We also saw that $H(z)$ is called as a minimum phase filter if all its poles and zeros are inside the unit circle.

To see the connection between a general $H(z)$ and its associated minimum phase and all-pass decomposition, let $H(z)$ be such that it has only one zero outside the unit circle.

$$H(z) = H_1(z)\,(z^{-1} - c_k^*)$$

That is, the zero is at $\dfrac{1}{c_k^*}$, where $|c_k| < 1$

Rewrite $H(z)$ as follows:

$$H(z) = H_1(z)(1-c_k z^{-1}) \cdot \frac{z^{-1}-c_k^*}{1-c_k z^{-1}}$$

Since $|c_k| < 1$, $H_1(z)(1-c_k z^{-1})$ is minimum phase and

$\frac{z^{-1}-c_k^*}{1-c_k z^{-1}}$ is all-pass.

This procedure can be repeated for every outside-unit-circle zero, and hence any $H(z)$ can be written as $H_{min}(z) \cdot H_{ap}(z)$

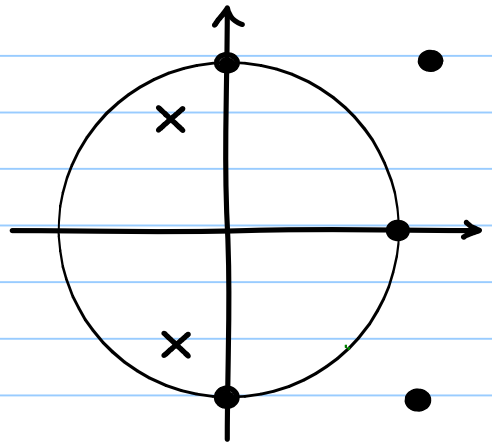
If $H(z)$ has zeros on the unit circle, then those zeros cannot be part of $H_{min}(z)$. Hence, the most general decomposition of $H(z)$ is as follows:

$$H(z) = H_{min}(z) \cdot H_{uc}(z) \cdot H_{ap}(z)$$

where $H_{uc}(z)$ contains all the unit circle zeros of $H(z)$.

If a system is minimum phase, causal, and stable, its inverse system is also causal, stable, and minimum phase.

$$\boxed{H(z) = H_{min}(z)\, H_{uc}(z)\, H_{af}(z)}$$



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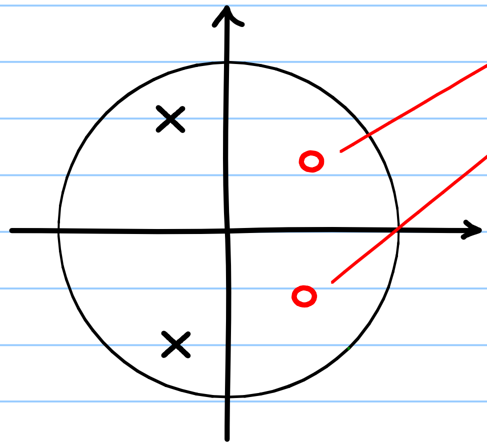
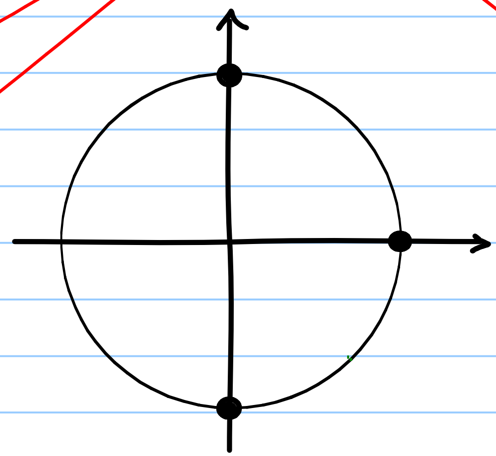
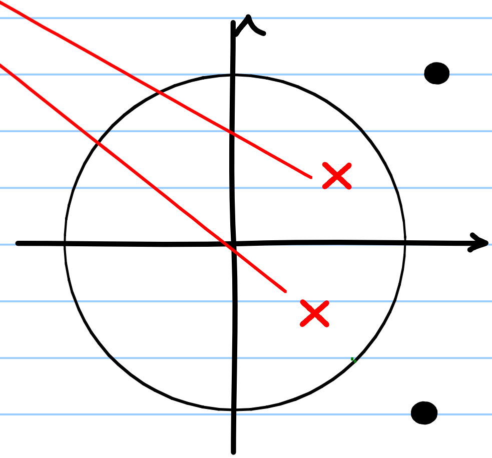


$H_{min}(z)$



$H_{uc}(z)$



$H_{af}(z)$