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Note Title 08-10-2013

Periodicity of $H(\omega)$, i.e., $H(\omega) = H(\omega+2\pi)$ and real-valuedness of the impulse response, i.e, $h[n] \in \mathbb{R}$, coupled with linear phase, impose some constraints. We will use the $H(\omega)$ notation rather than the usual $H(e^{j\omega})$.

Let $H(\omega) = A(\omega)e^{j(\beta-\omega\tau_g)}$

$$H(\omega+2\pi) = H(\omega)$$

i.e., $A(\omega)e^{j(\beta-\omega\tau_g)} = A(\omega+2\pi)e^{j(\beta-\omega\tau_g-2\pi\tau_g)}$

$$\Rightarrow \quad A(\omega) = A(\omega+2\pi)e^{-j2\pi\tau_g}$$

Since $A(\omega) \in \mathbb{R}$, $2\tau_g \in \mathbb{Z}$

(1) $\tau_g = \underbrace{M}_{\text{integer}} \Rightarrow A(\omega) = A(\omega + 2\pi)$ periodic with period 2π

(2) $\tau_g = \underbrace{M + \frac{1}{2}}_{\text{integer} + \frac{1}{2}} \Rightarrow A(\omega) = -A(\omega + 2\pi)$ periodic with period 4π

Also, since $h[n] \in \mathbb{R}$, $H(\omega) = H^*(-\omega)$. Hence,

$$A(\omega) e^{j(\beta - \omega \tau_g)} = A(-\omega) e^{-j(\beta + \omega \tau_g)}$$

$$\Rightarrow \quad \frac{A(\omega)}{A(-\omega)} = e^{-j2\beta}$$

Since $\frac{A(\omega)}{A(-\omega)} \in \mathbb{R}$, $\quad 2\beta = 0$ (or π) or $\frac{\pi}{2}$ (or $\frac{3\pi}{2}$)

(1) If $\beta = 0$, $A(\omega) = A(-\omega)$ Even symmetry

(2) If $\beta = \frac{\pi}{2}$ $A(\omega) = -A(-\omega)$ Odd symmetry

Thus, overall, we have FOUR possibilities:

$\tau_g = M$	$\beta = 0$	$A(\omega) = A(-\omega)$	Integer group delay
$\tau_g = M + \frac{1}{2}$	$\beta = 0$	$A(\omega) = A(-\omega)$	Integer $+\frac{1}{2}$ group delay
$\tau_g = M$	$\beta = \frac{\pi}{2}$	$A(\omega) = -A(-\omega)$	Integer group delay
$\tau_g = M + \frac{1}{2}$	$\beta = \frac{\pi}{2}$	$A(\omega) = -A(-\omega)$	Integer $+\frac{1}{2}$ group delay

Suppose we further assume that linear phase filter is CAUSAL.

$h[n] = 0$ for $n < 0$. First consider the case $\beta = 0$.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega) e^{-j\omega\tau_g} e^{j\omega n}\, d\omega$$

$$h^*[2\tau_g - n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\, e^{j\omega\tau_g}\, e^{-j\omega(2\tau_g - n)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\, e^{-j\omega\tau_g}\, e^{j\omega n}\, d\omega = h[n]$$

That is, for $\beta = 0$, $h[n] = h^*[2\tau_g - n]$ $\boxed{\text{for ANY linear phase filter.}}$

Hence, if we further assume $h[n] = 0$ for $n < 0$, $h^*[2\tau_g - n] = 0$ for $n > 2\tau_g$ $\Rightarrow$ $\boxed{\text{the filter is FIR}}$

For $\beta = \frac{\pi}{2}$, show that the condition to be satisfied is

$h[n] = -h^*[2\tau_g - n]$

$$\boxed{\begin{array}{l} h[n] = h^*[2\tau_g - n] \Rightarrow \text{symmetry around } n = \tau_g \\ h[n] = -h^*[2\tau_g - n] \Rightarrow \text{anti-symmetry around } n = \tau_g \end{array}}$$

Let the FIR filter be defined over the interval $n = 0, 1, \ldots, N-1$.

Hence $h[n] = 0$ for $n < 0$ and $n > N-1$. Hence $2\tau_g = N-1$, i.e.,

$\boxed{\tau_g = \frac{N-1}{2}}$. Therefore, if the length (N) of the filter is odd,

τ_g is an integer; if the length is even, τ_g equals

integer $+ \frac{1}{2}$ samples.

Therefore, the delay introduced by a linear phase FIR filter is either integer or integer $+ \frac{1}{2}$ samples.

Length	Symmetry	Group Delay	Name
Odd	Even $\beta = 0$	$\frac{N-1}{2}$ int	Type I
Even	Even $\beta = 0$	$\frac{N-1}{2}$ int $+ \frac{1}{2}$	Type II
Odd	Odd $\beta = \frac{\pi}{2}$	$\frac{N-1}{2}$ int	Type III
Even	Odd $\beta = \frac{\pi}{2}$	$\frac{N-1}{2}$ int $+ \frac{1}{2}$	Type IV