Note Title 07-10-2013

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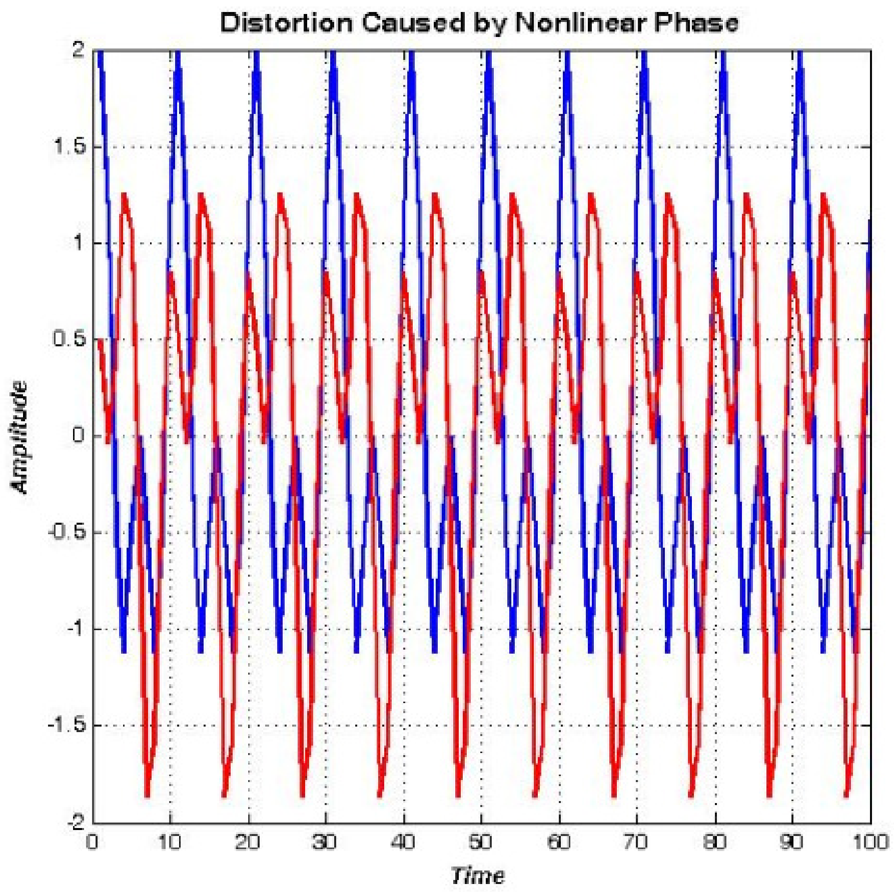
If the phase is not linear, waveshape will not be preserved. The blue curve is $\cos \omega_1 n + \cos \omega_2 n$. The red curve is $\cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2)$ where θ_i is not proportional to ω_i: $\boxed{\text{Waveshape is not preserved.}}$ If $\theta_i \propto \omega_i$, it will cause mere delay.



Linear phase with slope $-L$ will cause a delay of L samples.

The slope, in general, is not constrained to be an integer. What is the meaning of a slope that introduces non-integer delay? Assuming a sampling period T, a fractional delay of $L+\delta$ means that the output is the sampled version of the underlying continuous-time signal delayed by $(L+\delta)T$.

$$\text{If} \quad H(e^{j\omega}) = \begin{cases} e^{-j\omega(\overbrace{L+\delta}^{\text{non-integer}})} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

and the input $X(e^{j\omega})$ has components only in the passband, the output is given by

$$y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega(n-L-\delta)} d\omega$$

$$= \sum_{m=-\infty}^{\infty} x[m] \, \text{Sinc}(\overline{n-L-\delta} - m)$$

Group Delay

$$\tau_g(\omega) \triangleq -\frac{d}{d\omega}\phi(\omega)$$ where $\phi(\omega)$ is the continuous phase function

Phase Delay

$$\tau_p(\omega) \triangleq -\frac{\phi(\omega)}{\omega}$$

If $\phi(\omega) = -\alpha\omega$, then $\tau_g(\omega) = \tau_p(\omega) = \alpha$ [constant!]

$\tau_g(\omega)$ and $\tau_p(\omega)$ have the following interpretation:

If a narrowband signal is passed through a narrowband filter, the envelope of the output gets delayed by $\tau_g(\omega_0)$ and the carrier suffers a phase lag of $\tau_p(\omega_0)$, where ω_0 is the centre frequency.

Since $\phi(\omega) = \tan^{-1}\left[\dfrac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right]$, it is easy to see that

$$\tau_g(\omega) = \frac{H_I(e^{j\omega})\,H_R'(e^{j\omega}) - H_R(e^{j\omega})\,H_I'(e^{j\omega})}{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})} = -\,\mathrm{Im}\left\{\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right\}$$

For a single complex zero,

$$\phi(\omega) = \tan^{-1} \frac{r \, \mathrm{Sin}(\omega - \theta)}{1 - r \, \mathrm{Cos}(\omega - \theta)}$$

$$\Rightarrow \quad \tau_g(\omega) = \frac{r - \mathrm{Cos}(\omega - \theta)}{r + \frac{1}{r} - 2\mathrm{Cos}(\omega - \theta)}$$

If $r < 1$ and we replace $re^{j\theta}$ by $\frac{1}{r}e^{j\theta}$, then the above expression reveals that the group delay increases.

That is, reflecting an inside-unit-circle zero about the unit circle s.t. it now lies outside increases the group delay.

Units of $\tau_g(\omega)$ are samples

For real-valued $h[n]$, $\phi(\omega) = -\phi(-\omega) \Rightarrow \tau_g(\omega)$ is an even function.
[makes sense since delay at ω and $-\omega$ must be the same]

$\tau_g(\omega) > 0$ in the passband of causal, stable filters

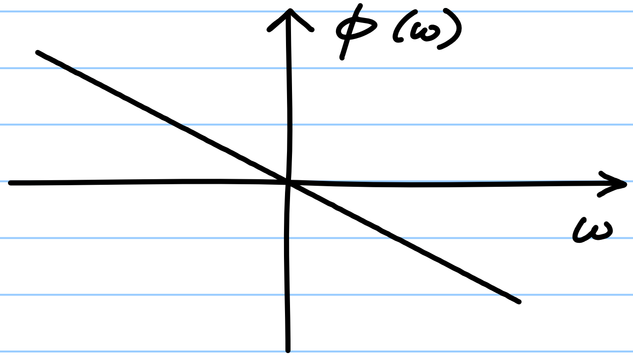
$\tau_g(\omega)$ can assume any real-value, not necessarily an integer.

In general, $\tau_g(\omega)$ is a nolinear function.

A rapid change in phase, typically caused by poles or zeros close to the unit circle, will cause a spike in $\tau_g(\omega)$.

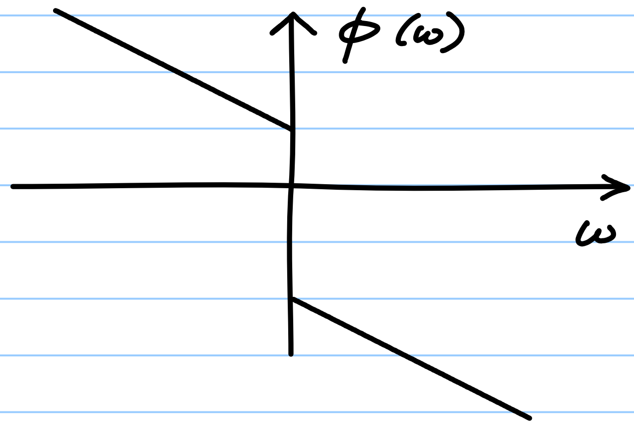
Since linear phase is essential for preserving waveshape, we will examine its consequences in more detail.

Linear Phase



$$\phi(\omega) = -\omega \tau_g$$

Generalized Linear Phase



$$\phi(\omega) = \beta - \omega \tau_g$$

Linear phase is a special case of generalized linear phase with $\beta = 0$.

For both case $\tau_g(\omega) = \tau_g$, a constant. But,

$$\tau_p(\omega) = \begin{cases} \tau_g & \text{for linear phase} \\ \tau_g - \dfrac{\beta}{\omega} & \text{for generalized linear phase} \end{cases}$$

What constraints, if any, are imposed on filters with linear phase?