Note Title 01-10-2013

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Define $W = \dfrac{w-1}{w+1} \Rightarrow w = \dfrac{1+W}{1-W}$

Since $Z = \dfrac{\eta - 1}{\eta+1}$, we get $\eta = \dfrac{1+Z}{1-Z}$

Recall that $w = \frac{1}{2}(\eta + \eta^{-1})$. Hence $\underbrace{\dfrac{w-1}{w+1}}_{W} = \underbrace{\left[\dfrac{\eta-1}{\eta+1}\right]^2}_{Z^2} \Rightarrow W = Z^2$

Therefore, $H(\eta) H(1/\eta) = H\left(\dfrac{1+Z}{1-Z}\right) H\left(\dfrac{1-Z}{1+Z}\right) = V(w) = V\left(\dfrac{1+W}{1-W}\right)$

Define $H\left(\dfrac{1+Z}{1-Z}\right) = H_1(Z) \qquad V_1(W) = V\left(\dfrac{1+W}{1-W}\right)$

Hence, $V_1(W) = H_1(Z)H_1(-Z) = V_1(Z^2)$

The above formulation is analogous to the spectral factorization problem of continuous-time systems with rational transfer function.

The steps for the alternate method are:

1) Replace $\cos w$ by $\dfrac{1+W}{1-W}$ to get $V_1(W)$

2) Find all the roots W_i of $V_1(W)$.

3) Form the eqn: $Z^2 = W_i \Rightarrow Z_i = \sqrt{W_i}$ and $-\sqrt{W_i}$

Z_i denotes the root with negative real part.

4) The poles/zeros of $H_1(Z)$ are the Z_i so obtained.

5) The unknown $H(\eta)$ equals $H_1\left(\frac{\eta - 1}{\eta + 1}\right)$.

The gain term is found from $H_1^2(0) = \mathcal{V}_1(0)$.

Example

$$A^2(\omega) = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega}$$

$$\mathcal{V}_1(W) = \frac{4 - 16W}{\frac{1}{4} - \frac{9}{4}W} \qquad W_1 = \frac{1}{4}, \quad W_2 = \frac{1}{9}$$

$$z_1^2 = \frac{1}{4} \Rightarrow z_1 = -\frac{1}{2}$$ (solution with negative real part)

$$z_2^2 = \frac{1}{9} \Rightarrow z_2 = -\frac{1}{3}$$

$$H_1(z_1) = K \cdot \frac{z + \frac{1}{2}}{z + \frac{1}{3}}$$

$$H_1^2(0) = K^2 (3/2)^2 = \tilde{V}_1(0) = 16 \Rightarrow K = \frac{8}{3}$$

$$H(y) = H_1\left(\frac{y-1}{y+1}\right) = \frac{8}{3} \frac{9y-3}{8y-4} = \frac{3y-1}{y-\frac{1}{2}}$$

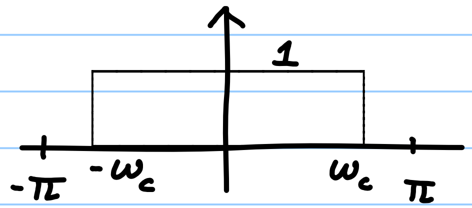
Note: By construction, min^m phase solution is obtained.

Group Delay

The phase response can be either strictly linear or nonlinear.

Suppose the frequency response is "zero phase", i.e., purely real-valued, then we need not bother about phase response.

Consider the ideal LPF.

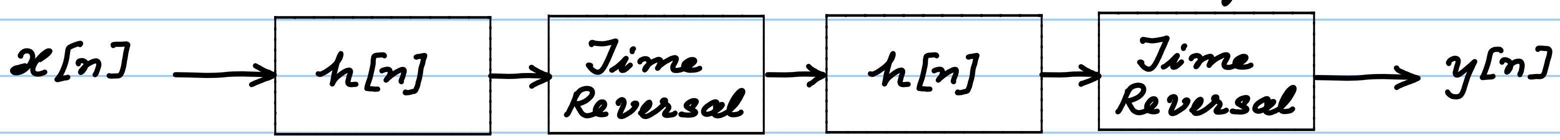


$$\longleftrightarrow \quad \frac{\sin \omega_c n}{\pi n}$$

The above filter is not realizable.

Suppose we approximate the ideal LPF using a rational transfer function, with frequency response shown below:

To realize a filter with "zero phase," assuming real-valued impulse response, consider the following sequence of operations:



It is easy to verify that $Y(e^{j\omega}) = X(e^{j\omega}) \underbrace{|H(e^{j\omega})|^2}_{\text{zero phase filter}}$

Unfortunately, the above sequence of operations results in a non-causal filter, and hence not realizable.

Instead of zero phase, if we had linear phase, the output of the linear phase filter will be a delayed version of zero phase filter's output. Although delay is a distortion in the strict sense, it is a benign one.

If rational transfer function approximations with linear phase are realizable, then they are what will be implemented in practice.

Let $\cos\omega_1 n + \cos\omega_2 n$ be an input to a filter.

Recall the following result:

$$\cos\omega_0 n \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow |H(e^{j\omega_0})| \cos\left(\omega_0 n + \angle H(e^{j\omega_0})\right)$$

If we want only delay distortion, i.e., output can, at the worst, only be a delayed version of the input, then

$$y[n] = x[n-\alpha] = \cos(\omega_0 \overline{n-\alpha})$$
$$= \cos(\omega_0 n - \omega_0 \alpha)$$

This means, $|H(e^{j\omega_0})| = 1$ and also $\measuredangle H(e^{j\omega_0}) = -\alpha\omega_0$

That is, the phase response must be proportional to frequency, apart from unity gain at that frequency.

When there are two components, for delay distortion,

$$y[n] = \cos(\omega_1 \overline{n-\alpha}) + \cos(\omega_2 \overline{n-\alpha})$$

$$= \cos(\omega_1 n - \omega_1 \alpha) + \cos(\omega_2 n - \omega_2 \alpha)$$

where once again the phase shift has to be proportional to frequency, i.e., linear.

Suppose a filter has gain $|H(e^{j\omega_i})| = 1$ for $i = 1, 2$, but the phase response is not linear. The output $y[n]$ will be

$$y[n] = \cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2)$$

where θ_i is not proportional to ω_i.

Will the waveshape be preserved ?