Note Title 28-08-2013

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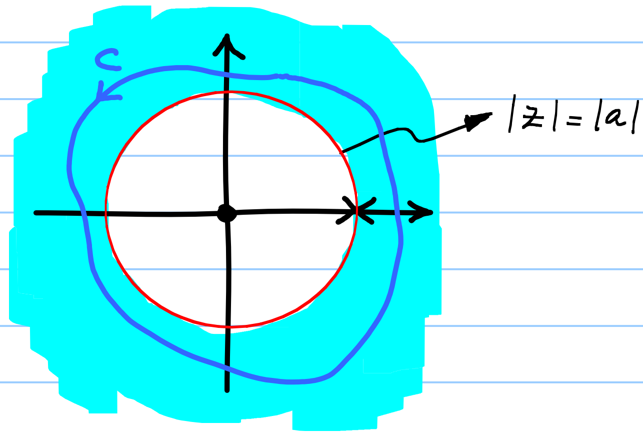
Example

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

$$= \frac{z}{z-a}$$



$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z}{z-a} z^{n-1} dz$$

$$= \frac{1}{2\pi j} \oint \frac{z^n}{z-a} dz$$

For $n \geq 0$, the contour encloses one pole at $z = a$

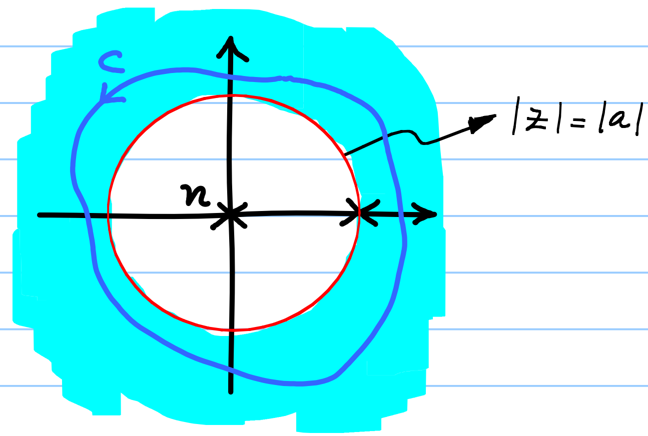
Residue at $z = a$: $\displaystyle (z-a)\frac{z^n}{z-a}\bigg|_{z=a} = a^n$

For $n < 0$, $\dfrac{z^n}{z-a}$ can be written as $\dfrac{1}{z^n(z-a)}$ where $n > 0$

Hence, one can now easily see that C encloses not only the pole at $z=a$ but also an n^{th} order pole at $z=0$.

Thus, residues have to be evaluated at $z=0$ and $z=a$.

Residue at $z=a$: $(z-a)\dfrac{z^n}{z-a}\Big|_{z=a} = a^n$ $\leftarrow n<0$



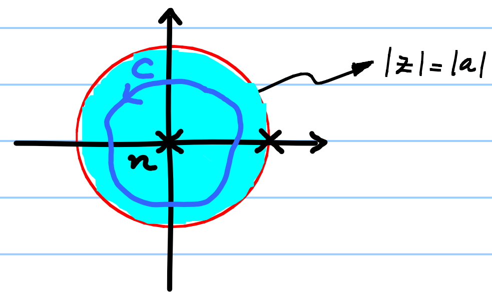
Residue at $z=0$: $\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\dfrac{1}{z-a}\Big|_{z=0}$

$$= -a^n$$

Hence, for $n<0$, sum of residues is zero.

Thus, $x[n] = a^n u[n]$

Repeat for $X(z) = \dfrac{1}{1-az^{-1}}$ with RoC $|z| < |a|$



Power Series Method

Examples:

(i) $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$

$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$

$\longleftrightarrow \{1, -\frac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{-1}, \frac{1}{2}\}$

(ii) $X(z) = \dfrac{1}{1-az^{-1}} \qquad |z| > |a|$

$= 1 + az^{-1} + a^2z^{-2} + \cdots$

$$\longleftrightarrow \{\underset{\substack{\uparrow \\ n=0}}{1}, a, a^2, \dots\}$$

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad a^3z^3 \\
 & \qquad\qquad \vdots
\end{array}$$

(iii) $X(z) = \dfrac{1}{1 - az^{-1}} \qquad |z| < |a|$

$$= \frac{-a^{-1}z}{1 - a^{-1}z}$$

$$= -a^{-1}z \cdot \left(1 + a^{-1}z + a^{-2}z^2 + \cdots\right)$$

$$= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots$$

$$\longleftrightarrow \{\dots, -a^{-3}, -a^{-2}, a^{-1}, \underset{\uparrow}{0}, 0, 0, \dots\}$$

$$\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots \\
\hline
-az^{-1} + 1 & 1 \\
 & \underline{1 - a^{-1}z} \\
 & \quad a^{-1}z \\
 & \quad \underline{a^{-1}z - a^{-2}z^2} \\
 & \qquad a^{-2}z^2 \\
 & \qquad \underline{a^{-2}z^2 - a^{-3}z^3} \\
 & \qquad\qquad a^{-3}z^3 \\
 & \qquad\qquad \vdots
\end{array}$$

(iv) $$X(z) = \frac{1-a^2}{(1+a^2) - a(z+z^{-1})}$$

If the RoC is $|a| < |z| < \frac{1}{|a|}$, then the corresponding $x[n]$ is $a^{|n|}$, i.e., it is two-sided.

If we carry out long-division directly, we will get a series expansion either in powers of z (anticausal sequence, corresponding to $|z| < |a|$) or in powers of z^{-1} (causal sequence, coresponding to $|z| > \frac{1}{|a|}$). We will not get the two-sided sequence.

To get the two-sided answer, we must proceed as follows:

$$X(z) = \frac{1}{1-az^{-1}} + \frac{az}{1-az}$$

causal part $|z|>|a|$; anticausal part $|z| < \frac{1}{|a|}$

Hence

$$X(z) = 1 + az^{-1} + a^2z^{-2} + \cdots \quad \text{causal part}$$
$$+ \; az + a^2z^2 + \cdots \quad \text{anticausal part}$$

$$\longleftrightarrow \{ \ldots, a^2, a, \underset{\uparrow}{1}, a, a^2, a^3, \ldots \}$$

(v) $X(z) = e^{z}$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad |z| < \infty$$ [can also be stated as $z \in \mathbb{C}$]

$$\longleftrightarrow \left\{ \ldots, \frac{1}{4!}, \frac{1}{3!}, \frac{1}{2!}, \frac{1}{1!}, \underset{\substack{\uparrow \\ n=0}}{1}, 0, 0, 0, \ldots \right\}$$

(vi) $\ln(1 + a z^{-1}) \qquad |z| > |a|$

Obtain the answer using both series expansion and the differentiation property.

The DTFT inversion formula can be derived from the z-transform inversion integral by substituting $z = e^{j\omega}$. The contour integral now becomes an integral over the real-valued variable 'ω'

$$z = e^{j\omega} \Rightarrow d\omega = \frac{dz}{jz}$$

Hence,

$$\boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega}$$ IDTFT

Recall, the DTFT definition:

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}} \quad \text{DTFT}$$

Since $X(e^{j\omega})$ is a 2π-periodic function, the DTFT can be thought of as the Fourier Series expansion of $X(e^{j\omega})$ with $x[n]$ as the Fourier series coefficients. Hence the DTFT is nothing but Fourier series in disguise.

Examples

(i) $X(e^{j\omega}) = 2\pi\,\delta(\omega) \qquad -\pi \leq \omega \leq \pi$

$$= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$ valid for all ω

$$= 2\pi\,\tilde{\delta}(\omega)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega)\,e^{j\omega n}\,d\omega$$

$$= 1$$

$$\therefore \quad 1 \xleftrightarrow{DTFT} 2\pi\,\tilde{\delta}(\omega)$$

(ii) $e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(\omega-\omega_0) \qquad -\pi \le \omega < \pi$

which also follows from the modulation property

(iii) $\cos \omega_0 n \xleftrightarrow{DTFT} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] \qquad -\pi \le \omega < \pi$

(iv) $\sin \omega_0 n \xleftrightarrow{DTFT} \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] \qquad -\pi \le \omega < \pi$

(v) $x[n] = 1 \quad -N \le n \le N \xleftrightarrow{DTFT} \dfrac{\sin (2N+1)\omega/2}{\sin(\omega/2)}$

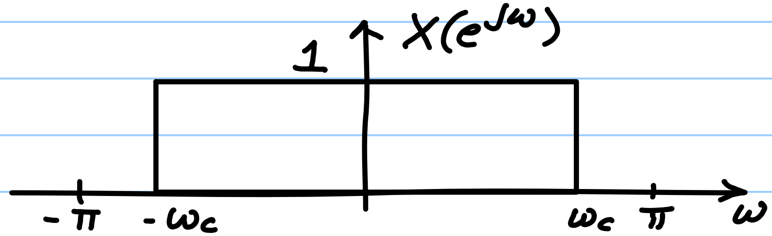
(vi) $X(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$

$$x[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.\ d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$

Hence,

$$\frac{\sin \omega_c n}{\pi n} \xleftrightarrow{\text{DTFT}}$$



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