EE 5330 Aug. 26, 2013

Note Title 26-08-2013

An alternative version of the Final Value Theorem:

Let the discrete-time signal $x[n]$ have the one-sided z-transform $X_+(z)$ defined as $\sum_{n=0}^{\infty} x[n] z^{-n}$. Then, if $\lim_{n \to \infty} x[n]$ exists,

$$\lim_{z \to 1} (z-1) X_+(z) = \lim_{n \to \infty} x[n]$$

Another variant: For a causal $x[n]$ s.t. $(z-1)X(z)$ can be analytically extended to $\{z : |z| > R\}$ with $R < 1$,

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$$

Example

$$x[n] = u[n] \longleftrightarrow \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$= \frac{z}{z-1}$$

Hence

$$\operatorname*{Lt}_{z \to 1} (z-1) \frac{z}{z-1} = 1 = x[\infty]$$

Note, however, that for $x[n] = (-1)^n u[n]$, $\operatorname*{Lt}_{z \to 1} (z-1) X(z) = 0$

which does not equal $x[\infty]$, as the latter limit does not exist.

11) <u>Parseval's Theorem</u>

Let $x[n] \longleftrightarrow X(z) \quad r_1^x < |z| < r_2^x$

$y[n] \longleftrightarrow Y(z) \quad r_1^y < |z| < r_2^y$

Then,

$$\sum_{n=-\infty}^{\infty} x[n]\, y^*[n] = \frac{1}{2\pi j} \oint_C X(z)\, Y^*(1/z^*) \frac{dz}{z}$$

$$r_1^x r_1^y < |z| = 1 < r_2^x r_2^y$$

For the corresponding DTFT property, let $z = e^{j\omega}$

$$dz = j\underbrace{e^{j\omega}}_{z}\,d\omega$$

$$\frac{dz}{z} = j\,d\omega$$

The contour integral now becomes a real-integral over ω; ω varies between $-\pi$ and π

Hence

$$\sum_{n=-\infty}^{\infty} x[n]\,y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,Y^*(e^{j\omega})\,d\omega$$

<u>Exercise</u>

Let $X(e^{j\omega}) = 1$ for $|\omega| < \omega_c$ and zero for $[-\pi, \pi) \setminus [-\omega_c, \omega_c]$

It can be shown that $x[n] = \dfrac{\sin \omega_c n}{\pi n}$

Using Parseval's Theorem, evaluate

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 \omega_c n}{\pi^2 n^2}$$

Inverse Z-Transform:

First consider the class of $X(z)$ that are rational, i.e., of the form

$$X(z) = \frac{P(z)}{Q(z)}$$

If the input-output relation of a system takes the form of a Linear Constant Coefficient Difference Equation, such as the one given below, then the system transfer function $H(z)$ is a rational one.

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{\ell=0}^{M} b_\ell \, x[n-\ell]$$

Taking z-transform on both sides,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{\ell=0}^{M} b_\ell z^{-\ell} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{\ell=0}^{M} b_\ell z^{-\ell}\right]$$

Hence,

$$\frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = H(z) = \frac{B(z)}{A(z)}$$ rational Transfer function

Associated with every LCCDE, there is a rational z-transform.

Conversely, with every rational z-transform, there is an associated LCCDE.

Since an LCCDE can be implemented in practice using multiplier and delay elements, the class of rational TFs is important. This class also models a lot of useful TFs.

Let $X(z) = \dfrac{P(z)}{Q(z)} = \dfrac{\sum_{\ell=0}^{M} p_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} q_k z^{-k}}$

$$= z^{N-M} \frac{\sum_{\ell=0}^{M} p_\ell z^{M-\ell}}{\sum_{k=0}^{N} q_k z^{N-k}} \quad \text{where } q_0 = 1$$

If $q_0 \neq 1$, we can always divide by q_0 so that the leading denominator coefficient is 1. Hence, without loss of generality, $q_0 = 1$ is assumed.

If $X(z) = \dfrac{\sum_{\ell=r}^{M} p_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} q_k z^{-k}}$, it can be written as

$$X(z) = z^{-r} \frac{P_1(z)}{Q(z)}$$ where there are no pole-zero cancellations.

The inverse z-transform of $\frac{P_1(z)}{Q(z)}$ and that of $\frac{P(z)}{Q(z)}$ differ only by a delay of 'r' samples.

Hence we will assume $p_0 \neq 0$ and $q_0 = 1$

First assume that all the roots are distinct

$$Q(z) = \prod_{k=1}^{N} (1 - q_k z^{-1})$$

$$X(z) = \frac{P(z)}{\prod_{k=1}^{N}(1-q_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1-q_k z^{-1}}$$

RESIDUE

(lookup the MATLAB command "residue")

$$A_k = \left. \frac{P(z)}{\prod_{\substack{l=1 \\ l\neq k}}^{N}(1-q_l z^{-1})} \right|_{z=q_k}$$

Example

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

$$= \frac{2}{1-z^{-1}} + \frac{-1}{1-\frac{1}{2}z^{-1}}$$

$\boxed{\text{To get the inverse z-transform, we need RoC information.}}$

Three choices:

(i) $|z| < 1/2$ (ii) $1/2 < |z| < 1$ (iii) $|z| > 1$

left-sided two-sided right-sided

(i) $-2u[-n-1] + (1/2)^n u[-n-1]$

(ii) $-2u[-n-1] - (1/2)^n u[n]$

(iii) $2u[n] - (1/2)^n u[n]$

The Final answer depends on which particular RoC is specified.