Note Title 19-08-2013

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Recall the exponential multiplication property:

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$\gamma^n x[n] \longleftrightarrow X(z/\gamma) \qquad |\gamma| r_1 < |z| < |\gamma| r_2$$

Let $y[n] = \gamma^n x[n]$

$$\Rightarrow \quad Y(z) = X(z/\gamma)$$

$$\Rightarrow \quad Y(\gamma z) = X(z)$$

Suppose $X(z) = \dfrac{P(z)}{Q(z)}$

$$Y(z) = \frac{P(z/\gamma)}{Q(z/\gamma)}$$

If z_0 is a zero of $X(z)$, i.e. $X(z_0) = 0 \Rightarrow P(z_0) = 0$

then $Y(\gamma z_0) = \frac{P(z_0)}{Q(z_0)} = 0 \Rightarrow$ γz_0 is a zero of $Y(z)$

Similarly, if z_1 is a pole of $X(z)$, i.e., $Q(z_1) = 0$

then $Y(\gamma z_1) = \frac{P(z_1)}{Q(z_1)} \rightarrow \infty \Rightarrow$ γz_1 is a pole of $Y(z)$

All poles and zeros get multiplied by γ

Geometrically, each pole/zero gets scaled by $|\gamma|$ and rotated by $\measuredangle \gamma$.

4) Differentiation in the z-domain

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$? \longleftrightarrow -z \frac{dX}{dz} \qquad \text{RoC } ?$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x[n] z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{d}{dz} z^{-n}$$

This operation is allowed because the power series is absolutely convergent in the RoC

$$= \sum_{n=-\infty}^{\infty} (-n)\, x[n]\, z^{-n-1}$$

$$-z \frac{dX}{dz} = \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

Hence, $n\, x[n] \longleftrightarrow -z \dfrac{dX(z)}{dz}$

Since $X(z)$ is analytic in the RoC, it can be differentiated infinite no. of times. Hence, the above property can be repeatedly applied

The RoC of $-z\dfrac{dX}{dz}$ is the same as the RoC of $X(z)$ except possibly for the deletion of the boundary circle (if it were part of the original RoC)

Example

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-z\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{(-z)(-1)(-a)(-z^{-2})}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

i.e., $$n a^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1) a^{n+1} u[n+1] \longleftrightarrow \frac{a}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1)\, a^{n}\, u[n+1] \longleftrightarrow \frac{1}{(1-az^{-1})^{2}} \qquad |z| > |a|$$

Can be rewritten as,

$$(n+1)\, a^{n}\, u[n] \longleftrightarrow \frac{1}{(1-az^{-1})^{2}} \qquad |z| > |a|$$

Repeat the above steps by starting with $\frac{1}{1-az^{-1}}$ but with RoC $|z| < |a|$. At what index does the first non-zero sample begin?

5) Complex Conjugation

$$x^*[n] \longleftrightarrow X^*(z^*) \qquad r_1 < |z| < r_2$$

$$\sum_{n=-\infty}^{\infty} x^*[n]\, z^{-n} = \left[\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right]^*$$

$$= X^*(z^*) \qquad r_1 < |z| < r_2$$

The corresponding property for the DTFT is:

$$X^*(z^*)\Big|_{z=e^{j\omega}} = X^*(e^{-j\omega})$$

If $x[n] \in \mathbb{R}$, then $x^*[n] = x[n]$

Hence, for real-valued sequences, the z-transform satisfies

$X(z) = X^*(z^*)$

For such sequences, if z_0 is a zero of $X(z)$, then $X(z_0) = 0$.

Therefore, $X(z_0) = 0$

$\Rightarrow X(z_0) = X^*(z_0^*)$

$\Rightarrow X^*(z_0^*) = 0$

$\Rightarrow X(z_0^*) = 0$

$\Rightarrow z_0^*$ is also a zero of $X(z)$

Thus, zeros occur in complex conjugate pairs.
Similarly, it is easy to see that poles also occur in complex conjugate pairs.

Also, for real-valued sequences, $X(e^{j\omega}) = X^*(e^{-j\omega})$ [conjugate even]

$$\Rightarrow |X(e^{j\omega})| = |X^*(e^{-j\omega})|$$

$$= |X(e^{j\omega})|$$ DTFT mag. is an even function of ω

Exercise

Starting from $X(e^{j\omega}) = X^*(e^{-j\omega})$, show that

$\angle X(e^{j\omega})$ is an odd function of ω

6) <u>Time Reversal</u>

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$x[-n] \longleftrightarrow X(z^{-1}) \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

This operation makes a causal sequence non-causal and vice-versa

<u>Example</u>

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

Using the time-reversal property,

$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-az} \qquad |z| < \frac{1}{|a|}$$

$$\frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$a^{-n}u[-n] \longleftrightarrow \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$a^{-n-1}u[-n-1] \longleftrightarrow \frac{-a^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$-a^{-n}u[-n-1] \longleftrightarrow \frac{1}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Let $b = a^{-1}$. Hence,

$$-b^{n}u[-n-1] \longleftrightarrow \frac{1}{1-bz^{-1}} \qquad |z| < |b|$$

as before.