EE 5330 Aug. 13, 2013

Note Title 13-08-2013

Properties of the Z-Transform:

1) Linearity

$$y[n] = a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{Z} a_1 X_1(z) + a_2 X_2(z)$$

$$RoC_y \supseteq RoC_{x_1} \cap RoC_{x_2}$$

The RoC is atleast as large as the intersection of the two RoCs, but can be larger if there are some pole-zero cancellations.

$$y[n] = a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{DTFT} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

$$x_1[n] = a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z|>|a|$$

$$x_2[n] = a^n u[n-N] \longleftrightarrow \frac{a^N z^{-N}}{1-az^{-1}} \qquad |z|>|a|$$ (See next property for derivation)

$$x_1[n] - x_2[n] \longleftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}} = 1 + az^{-1} + \cdots + a^{N-1} z^{-(N-1)}$$

The RoC is $|z|>0$ and larger than $|z|>|a|$ because the pole at $z=a$ gets cancelled.

2) <u>Time Shift</u>

$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$$ RoC is identical except possibly for the addition or deletion of 0 and/or ∞

$$y[n] = x[n-n_0] \xleftrightarrow{DTFT} Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Note that $|Y(e^{j\omega})| = |X(e^{j\omega})|$

$x[n] = 1 \quad -N \leq n \leq N$

$$X(z) = z^N + z^{N-1} + \cdots + z + 1 + z^{-1} + \cdots + z^{-N}$$

$$= \frac{z^N(1 - z^{-\overline{2N+1}})}{1 - z^{-1}} \qquad 0 < |z| < \infty$$

$$= \frac{z^N - z^{-N-1}}{1 - z^{-1}} = \frac{z^{N+\frac{1}{2}} - z^{-N-\frac{1}{2}}}{z^{1/2} - z^{-1/2}}$$

$$y[n] = x[n-N] \Rightarrow Y(z) = z^{-N} X(z)$$

$$Y(z) = \frac{1 - z^{-(2N+1)}}{1 - z^{-1}} \qquad |z| > 0$$

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$

$$= \frac{e^{j\omega(N+1/2)} - e^{-j\omega(N+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin\left[(2N+1)\,\omega/2\right]}{\sin(\omega/2)}$$

Dirichlet kernel

[See the command 'diric' in MATLAB]

$$Y(e^{j\omega}) = e^{-j\omega N}\,\frac{\sin\left[(2N+1)\,\omega/2\right]}{\sin(\omega/2)}$$

Transform of $a^n u[n-N]$ can be easily obtained using the delay property

$$a^N \overbrace{a^{n-N} u[n-N]}^{x[n-N]} \longleftrightarrow a^N \frac{z^{-N}}{1-az^{-1}}$$

3) Exponential Multiplication

$$\gamma^n x[n] \longleftrightarrow X(z/\gamma) \qquad |z/\gamma| \in RoC_x$$

$$u[n] \longleftrightarrow \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-(z/a)^{-1}} \qquad |z/a| > 1$$

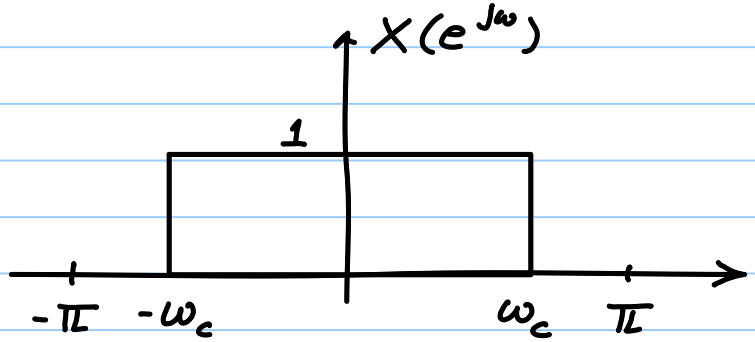
$$= \frac{1}{1-az^{-1}} \qquad |z| > |a|$$ as before

$$e^{j\omega_0 n} x[n] \overset{z}{\longleftrightarrow} X(z/e^{j\omega_0})$$

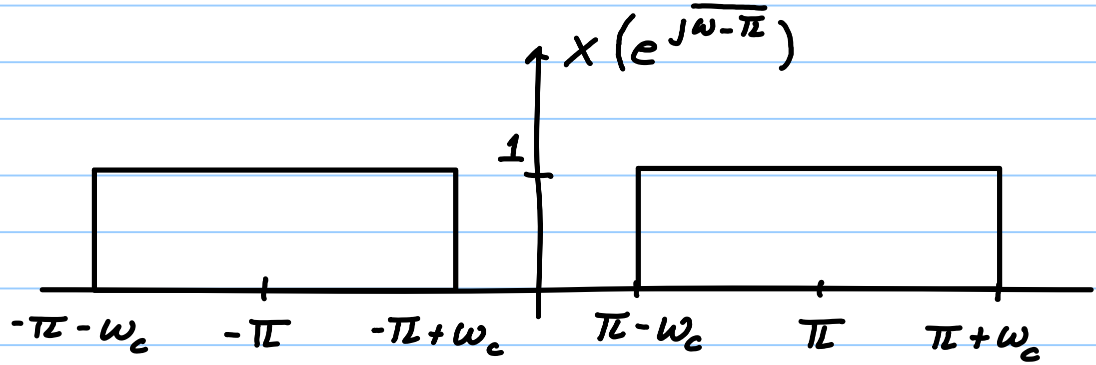
$$e^{j\omega_0 n} x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}/e^{j\omega_0}) = X(e^{j\overline{\omega-\omega_0}})$$ Modulation property !

$$(-1)^n x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\overline{\omega-\pi}}) = X(e^{j\overline{\omega+\pi}})$$

$$(-1)^n x[n] \overset{z}{\longleftrightarrow} X(-z)$$

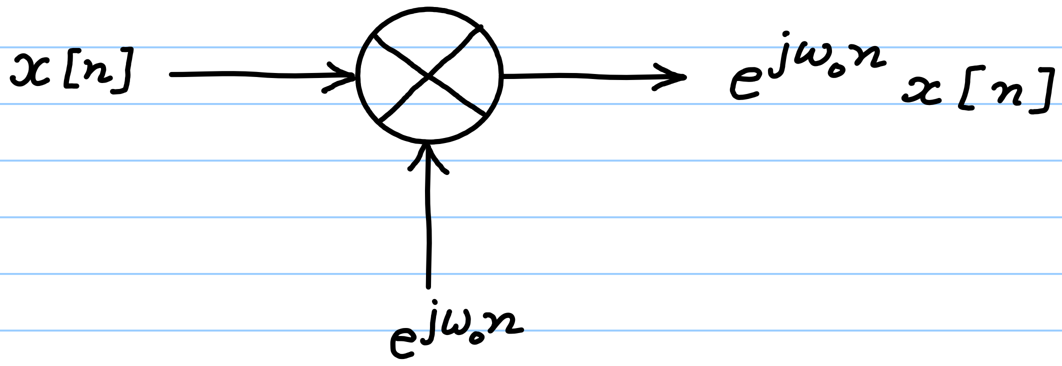


lowpass



highpass

Modulator block diagram:



Is the modulator a linear system?

Is it time-invariant?

Using the exponential multiplication property, derive the following:

$$r^n \cos \omega_0 n \, u[n] \xleftrightarrow{z} \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}} \qquad |z| > r$$

Plot the poles and zeros

Hint: Replace $\cos \omega_0 n$ by $\dfrac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$