Note Title 08-08-2013

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Isolated Singular Point:

z_0 is a singular point of $X(z)$ if it fails to be analytic at z_0. A singular point is isolated if, in addition, there is a deleted neighbourhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which X is analytic.

Example

$$X(z) = \frac{z+1}{z^3(z^2+1)}$$

Singularities are at $z = 0, \pm j$

Example

$$X(z) = \frac{1}{\sin(\pi/z)}$$

Singular points: $z = 0$ and $z = \frac{1}{n}$ $n = \pm 1, \pm 2, \ldots$

all lying on the real axis from $z = -1$ to $z = 1$.

Singularities at $z = \frac{1}{n}$ are isolated.

Singularity at $z = 0$ is not isolated because any ε-neighbourhood around $z = 0$ will contain other singularities.

Three Types of Singularities

Recall

$$X(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{"principal part"}}$$

where the expansion is in a punctured disk $0 < |z - z_0| < R$

If $b_m \neq 0$ but $b_{m+1} = b_{m+2} = \cdots = 0$, then the singularity at $z = z_0$ is a **pole of order m**.

Example

$$X(z) = \frac{z^2 - 2z + 3}{z - 2} = z + \frac{3}{z-2}$$

$$= 2 + (z-2) + \frac{3}{z-2} \qquad 0 < |z-2| < \infty$$

Hence we conclude that there is pole of order 1 at $z = 2$.

Example

$$X(z) = \frac{1}{z^2(1+z)}$$

$$= \frac{1}{z^2}\left(1 - z + z^2 - z^3 + \cdots\right) \qquad 0 < |z| < 1$$

$$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots \qquad 0 < |z| < 1$$

Hence $X(z)$ has a pole of order 2 at $z = 0$.

In the above two examples, the order and location of the poles can be inferred from the expression for $X(z)$ directly, without going through the series expansion.

This is because $X(z)$ was a rational function, i.e., a ratio of polynomials in z, of the form $P(z)/Q(z)$

The power of the series expansion method is made clear in the following example.

<u>Example</u>

$$X(z) = \frac{\sinh z}{z^4}$$

$$= \frac{1}{z^4}\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right] \qquad 0 < |z| < \infty$$

$$= \frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots \qquad 0 < |z| < \infty$$

There is a pole of order 3 at $z = 0$.

Example

$$X(z) = \frac{1 - \cos z}{z^2}$$

$$= \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] \qquad 0 < |z| < \infty$$

$$= \frac{1}{2!} + \frac{z^2}{2!} - \frac{z^2}{4!} + \frac{z^4}{6} - \cdots \qquad 0 < |z| < \infty$$

If we define $X(0) = \frac{1}{2}$, then $X(z)$ has no singularities.

Thus, $X(z) = \frac{1 - \cos z}{z^2}$ has a removable singularity.

Removable Singularity

If $X(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ over $0 < |z - z_0| < R$, then $X(z)$ has a removable singularity at $z = z_0$. If we now define $X(z_0) = a_0$, $X(z)$ becomes ENTIRE, i.e., analytic over the entire z-plane.

Essential Singularity

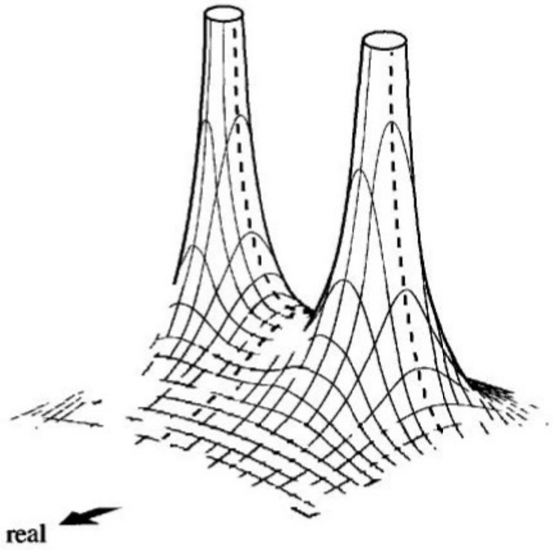
If the series representation of $X(z)$ over the punctured disk $0 < |z - z_0| < R$ contains all negative powers of $z - z_0$, then $z = z_0$ is an essential singularity.

Example

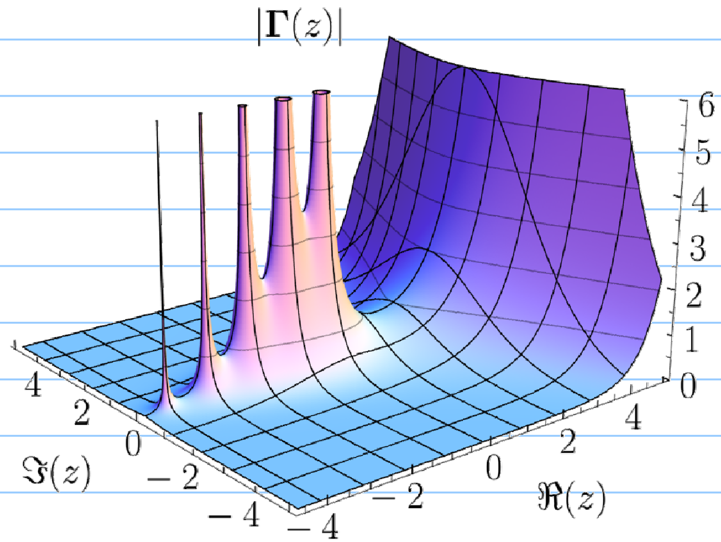
$$X(z) = e^{1/z}$$

$$= 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \cdots \qquad 0 < |z| < \infty$$

$z = 0$ is an essential singularity.



From *Visual Complex Analysis* by T. Needham, Oxford University Press, 1999, p. 66.



The absolute value of the Gamma function. This shows the function becomes infinite at the poles at n = -1, -2, -3, ... (Wikipedia, "Pole (complex analysis)")

An isolated singular point z_0 of a function $X(z)$ is a pole of order m if and only if $X(z)$ can be written in the form

$$X(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0