Note Title 07-08-2013

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$x[n] = \frac{1}{n} \quad n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty \Rightarrow e^{j\omega} \notin \text{RoC}$

$\text{RoC} : |z| > 1$

<u>Example</u>

$x[n] = \frac{1}{n^2} \quad n = 1, 2, \ldots$

$e^{j\omega} \in \text{RoC}$ can show $\text{RoC} : |z| \geq 1$

The point that we wish to illustrate with the above example is that the inequality need not always be strict.

However, for the major class of Z-transforms that we encounter in this course, the inequality will be strict.

The Z-transform is analytic in the region of convergence. An analytic function satisfies the

Cauchy-Riemann equations. That is, if

$$f(x+jy) = u(x,y) + j\,v(x,y)$$

is analytic, then

$$u_x = v_y$$

$$u_y = -v_x$$

where $u_x \equiv \frac{\partial}{\partial x} u(x,y)$ and so on.

Satisfying the CR equations alone does not guarantee analyticity. What is needed is the following.

Looman-Menchoff Theorem

Let $f = u + jv$ be defined on a domain D such that

(i) f is continuous on D

(ii) u_x, u_y, v_x, v_y exist everywhere on D (but not necessarily continuous)

(iii) u and v satisfy the CR equations.

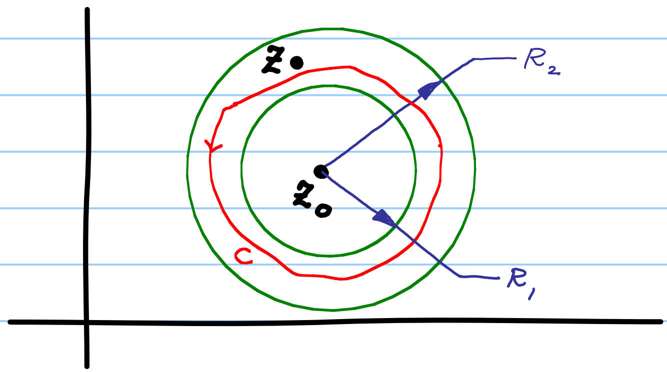
Then f is holomorphic on D. The term analytic is also used interchangeably

The z-transform $X(z)$ of $x[n]$ is analytic in the RoC.
The RoC cannot contain singularities.

The z-transform is in the form of a Laurent Series, whose definition is given below:

Laurent Series

Suppose f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$ centred at z_0. C is as shown in the figure. Then, at each point in the domain, $f(z)$ has the representation



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$R_1 < |z| < R_2$$

where

$$a_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz \qquad n = 1, 2, 3, \ldots$$

The above can be combined into a single expression:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad R_1 < |z - z_0| < R_2$$

where

$$c_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \qquad n = 0, \pm 1, \pm 2, \ldots$$

Examples

$$X(z) = e^{z}$$

$$= \sum_{n=0}^{\infty} \frac{z^{n}}{n!} \qquad |z| < \infty$$

$$X(z) = e^{1/z}$$

$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \qquad 0 < |z| < \infty$$

<u>Example</u>

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

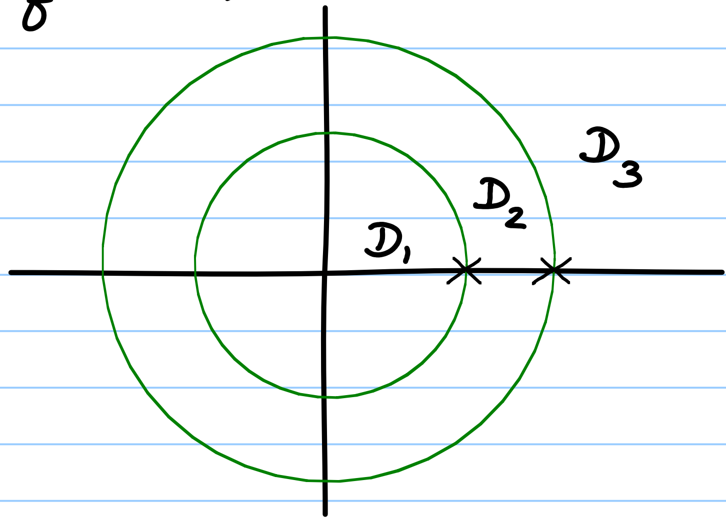
The singularities in $X(z)$ are at $z=1$, $z=2$.

Consider the series expansion of $X(z)$ in 3 different regions.

D_1 : $|z| < 1$

D_2 : $1 < |z| < 2$

D_3 : $2 < |z| < \infty$



In D_1, i.e., $|z| < 1$, $|z/2| < 1$

$$X_1(z) = \frac{-1}{1-z} + \frac{1}{2}\,\frac{1}{1-z/2} \qquad |z| < 1$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \qquad |z|<1$$

$$= \sum_{n=0}^{\infty}\left(2^{-n-1} - 1\right) z^n \qquad |z|<1$$

The above expansion contains only +ve powers of z.

In D_2, i.e., $1<|z|<2$, $\left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{2}\right|<1$. Hence,

$$X_2(z) = \frac{1}{z}\,\frac{1}{1-1/z} + \frac{1/2}{1-z/2} \qquad 1<|z|<2$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \qquad 1<|z|<2$$

The above expansion contains both +ve and -ve powers of z.

In D_3, i.e., $2 < |z| < \infty$, $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{2}{z}\right| < 1$. Hence,

$$X_2(z) = \frac{1}{z}\frac{1}{1-\frac{1}{2}} + \frac{1}{z}\frac{1}{1-\frac{2}{z}} \qquad 2 < |z| < \infty$$

$$= \sum_{n=1}^{\infty} z^{-n} + \frac{1}{z}\sum_{n=0}^{\infty} \frac{2^n}{z^n} \qquad 2 < |z| < \infty$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \qquad 2 < |z| < \infty$$

The above expansion contains only -ve powers of z.

Thus $X(z)$ has 3 different series expansions in the 3 different regions. Each series expansion is valid only in one particular region.

$$X_1(z) = X(z) \text{ in } D_1 \text{ i.e., } X_1(z) = X(z)\Big|_{D_1}$$

$$X_2(z) = X(z) \text{ in } D_2 \text{ i.e., } X_2(z) = X(z)\Big|_{D_2}$$

$$X_3(z) = X(z) \text{ in } D_3 \text{ i.e., } X_3(z) = X(z)\Big|_{D_3}$$