EE 5330 DSP Lecture 1, 29.07.13

Review basic discrete-time signals and systems

Refer to standard textbooks such as Lathi or Oppenheim, et al.

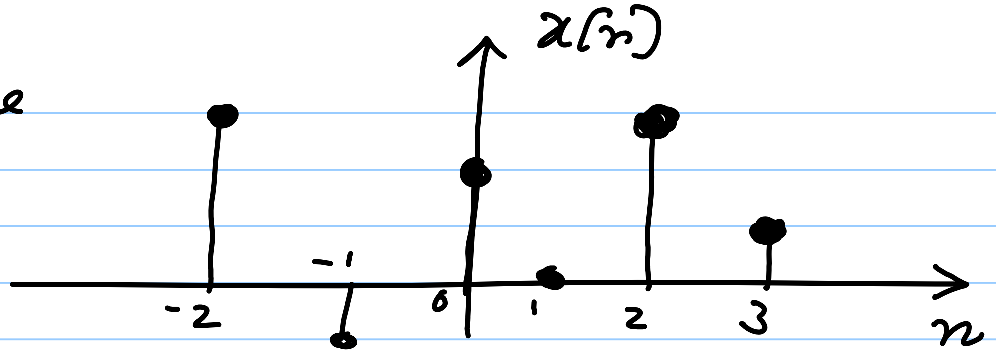
Discrete-Time Sequence:

$$x[n] \in \mathbb{C} \qquad n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$$

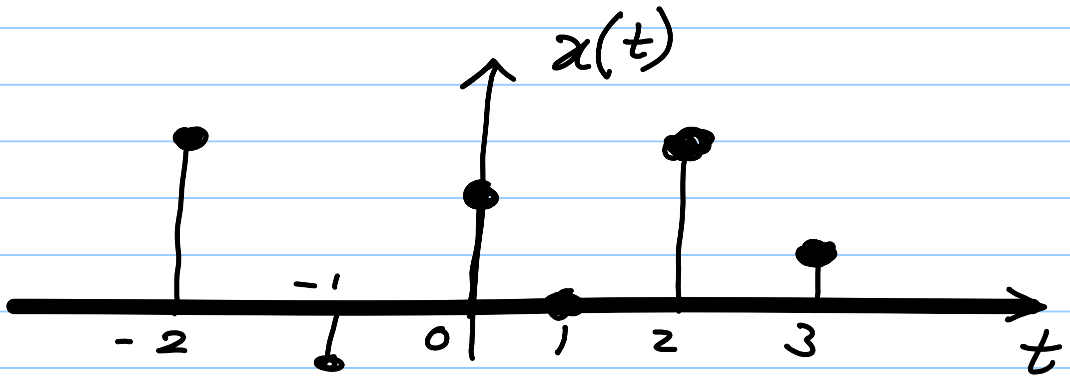
$x[a]$ = undefined if $a \notin \mathbb{Z}$

Note that "undefined" is not the same as saying it is zero.

A discrete-time sequence :



What's wrong with the following framework for DT signals?



$x(t) \in \mathbb{C}$ if $t \in \mathbb{Z}$

$x(t) = 0$ if $t \notin \mathbb{Z}$

Why do we need the new discrete-time framework introduced earlier ?

Review the basic discrete-time signals

Exponential class is an important class

i.e., $x[n] = z_0^n$ where $z_0 \in \mathbb{C}$

Note that z_0^n defined over all 'n' is an everlasting exponential. It is not the same as

$$x[n] = \begin{cases} z_0^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$x[n] = e^{jw_0 n}$ is a complex sinusoid

$x[n] = x[n+N]$ iff $\frac{\omega_0}{2\pi} = \frac{k}{N}$

There can be only N distinct complex exponentials with period N, corresponding to $k = 0, 1, \ldots, N-1$.

Recall & understand the differences between DT and CT complex sinusoids.

Does a DT sinusoid's rapidity of oscillations keep on increasing with increase in frequency?

In the CT case, the period of the k-th harmonic is k times smaller than the fundamental.

Is the same true of the DT harmonics also?

Are the sinusoids $e^{j\omega_0 n}$ & $e^{-j\omega_0 n}$ independent?

That is, if $a_1 e^{j\omega_0 n} + a_2 e^{-j\omega_0 n} = 0$, does it mean that $a_1 = a_2 = 0$ is the only solution?

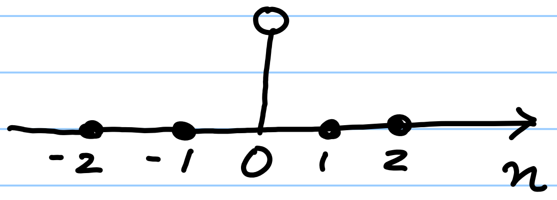
EE 5330 30th Jul., 2013

$e^{j\omega_0 n}$ & $e^{-j\omega_0 n}$ are independent

i.e., $\nexists\ k \in \mathbb{C}$, s.t. $e^{j\omega_0 n} = k\, e^{-j\omega_0 n}$

$u[n]$: unit step (cf. with $u(t)$)

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

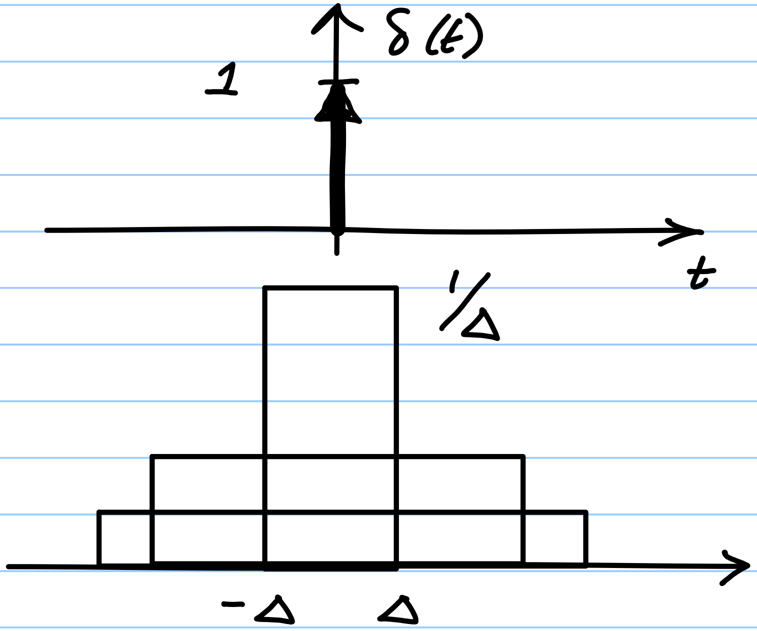


Cf. this with $\delta(t)$

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$\delta(t) + \delta'(t)$ also satisfies these two equations



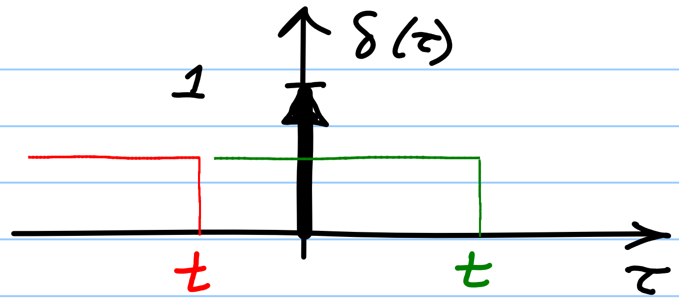
$\delta(0) = \infty$ wrong!

$\delta(0) = 1$ wrong!

$\delta(t)$ is a shorthand for limiting arguments

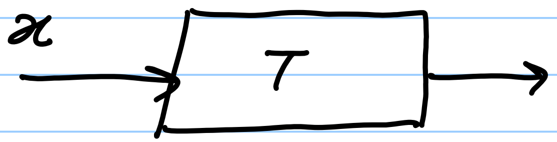
See *The Theory of Distributions: A Nontechnical Introduction* by J. Ian Richards and Heekyung K. Youn, Cambridge University Press (1995)

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



$$\frac{d}{dt} u(t) = \delta(t)$$

Systems



$y = Tx$

(a) Linearity : $a_1 x_1 + a_2 x_2 \longrightarrow a_1 y_1 + a_2 y_2$

(i) additivity (ii) homogeneity

These are two independent properties. (i) + (ii) ⇒ linear

$$\sum_{k=1}^{N} a_k x_k \longrightarrow \sum_{k=1}^{N} a_k y_k$$

What about

$$\sum_{k=1}^{\infty} a_k x_k \longrightarrow \sum_{k=1}^{\infty} a_k y_k$$

Follows if the space is complete

(b) Time Invariance $T\{x[n-n_0]\} = y[n-n_0]$

(c) Causality : O/p at $n = n_0$ depends only on i/p for time $n \leq n_0$

(d) Systems w/ memory : If o/p at $n = n_0$ depends only on i/p at $n = n_0$, the system is memoryless.

(e) Stability BIBO : bounded i/p → bounded o/p

If $|x[n]| < B_x < \infty$, then $T\{x[n]\} = y[n]$

is s.t. $|y[n]| < B_y < \infty$

We will focus on the class of Linear & Time Invariant (LTI) systems in this course.

EE 5330 31st Jul., 2013

Among LTI systems, we focus on

$$y[n] = F\{x[n], x[n-1], \ldots, x[n-M], y[n-1], y[n-2], \ldots, y[n-N]\}$$

In particular, we consider the class

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{l=0}^{M} b_l\, x[n-l]$$

i.e., system represented by LCCDE.

For an LTI system, the impulse response completely characterizes the system.

The general conditions of causality, stability, etc. can be translated into conditions on the impulse response.

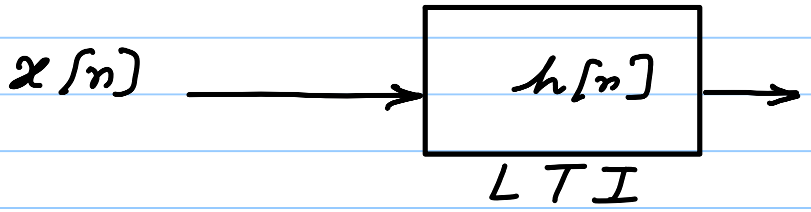
E.g.:

Causality: $h[n] = 0$ for $n < 0$

Stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

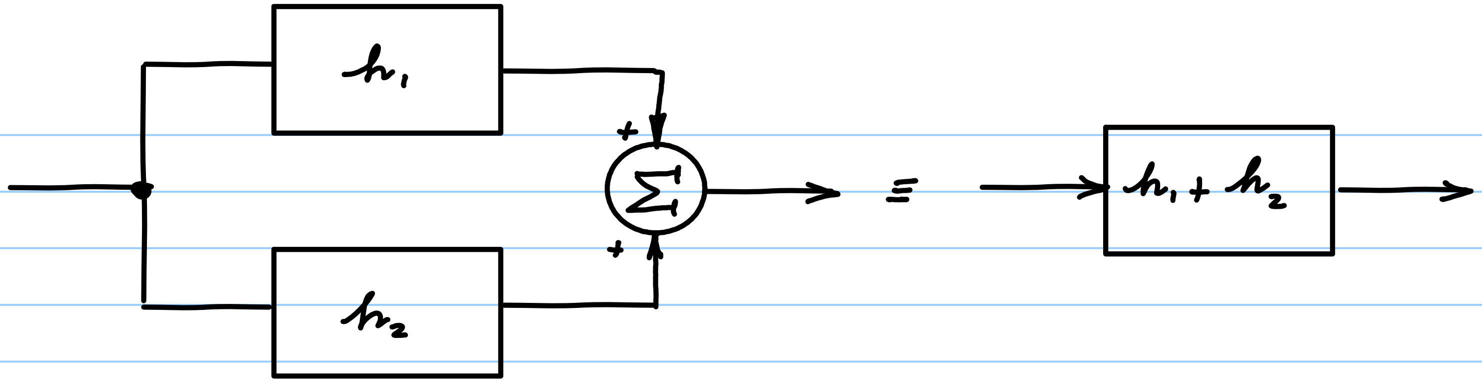
For an LTI system, knowing the impulse response means complete knowledge of the system.

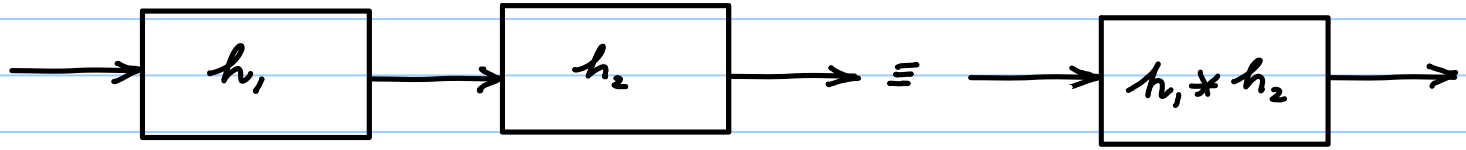
Linearity and time invariance imply that the o/p to any input is given by the following convolution sum:



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Parallel and cascade decomposition of systems are useful in Digital Filter implementation.





$x * (y + z) = x * y + x * z$ distributive

$x * (y * z) = (x * y) * z$ associative

Associativity holds if $x, y, z \in \ell_1$

l_1 is the space of absolutely summable sequences:

$$l_1 = \left\{ x : \sum_{n=-\infty}^{\infty} |x[n]| < \infty \right\}$$

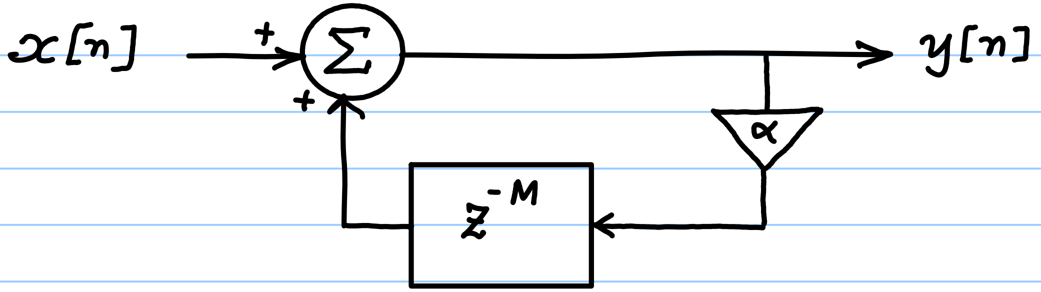
l_2 is the space of square summable sequences:

$$l_2 = \left\{ x : \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \right\}$$

Is $l_1 \subset l_2$? That is, if $x \in l_1$, does it also belong to l_2 ? Can you find y s.t. $y \in l_2$ but $y \notin l_1$?

A simple difference equation:

$$y[n] = \alpha\, y[n-M] + x[n]$$



Let $M = 100$, $\alpha = 0.98$ and $x[n] = \begin{cases} r & 0 \leq n \leq 99 \\ 0 & n \geqslant 100 \end{cases}$

where $r \in \mathcal{U}[-1, 1]$

Sounds like a plucked string when played at $F_s = 44.1$ kHz!

Take a look at the Karplus-Strong algorithm for more details

EE5330 Aug. 1, 2013

Systems → Finite Impulse Response (FIR)
Systems → Infinite Impulse Response (IIR)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

If the system is IIR, then at least one a_k is non-zero

If the system is FIR, then $a_k = 0$ for $k = 1, 2, \ldots, N$.

Consider the following system:

$$y[n] = \frac{1}{N}\sum_{k=n-N+1}^{n} x[n] \quad \text{———} \quad \textcircled{1}$$

(FIR system)

The above is the same as

$$y[n] = y[n-1] + \frac{1}{N}x[n] - \frac{1}{N}x[n-N] \quad \text{——} \quad \textcircled{\text{II}}$$

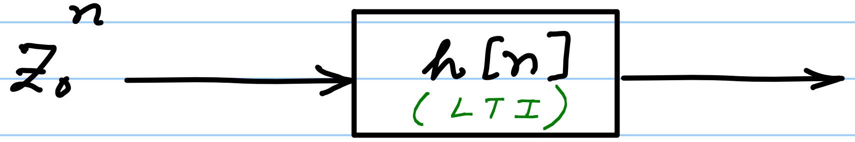
Eqn, $\textcircled{\text{II}}$ is a recursive implementation of a non-recursive equation, i.e., even though a_1 is non-zero, the system is not IIR

Z-Transform

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

One way of arriving at this is by applying the eigensignal z_0^n as the input to an LTI system:



$$y[n] = z_0^n * h[n]$$

$$= \sum_{k=-\infty}^{\infty} z_0^{n-k} h[k]$$

$$= z_0^{n} \sum_{k=-\infty}^{\infty} h[k]\, z_0^{-k}$$

$$= z_0^{n}\, H(z_0)$$

where $$H(z_0) = \sum_{k=-\infty}^{\infty} h[k]\, z_0^{-k}$$

The z-transform is a complex function of a complex variable and hence requires 4 dim. to plot: 2 for the indep. var. and 2 for the dep. variable

Refer to CONFORMAL MAPPING

Examples: $x[n] = a^{n} u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^{n} z^{-n}$$

$$= \frac{1}{1-az^{-1}} \quad \text{if } |az^{-1}| < 1$$

$$\text{i.e., } |z| > |a|$$

Example

$$x[n] = -a^{n} u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^{n} z^{-n}$$

$$= \frac{1}{1-az^{-1}} \quad \text{if. } |z| < |a|$$

The algebraic expression for X(z) is the same as before but the region over which it is valid is different

The range of 'z' over which the z-transform expression is valid is called Region of Convergence (RoC)

Examples

$$\{ \underset{\uparrow}{-1}, 2, 4, \pi \} \longleftrightarrow -1 + 2z^{-1} + 4z^{-2} + \pi z^{-3} \qquad 0 \notin RoC$$

right sided, $n \geq 0$

$$\{ -1, \underset{\uparrow}{2}, 4, \pi \} \longleftrightarrow -z + 2 + 4z^{-1} + \pi z^{-2} \qquad 0, \infty \notin RoC$$

two-sided

$$\{ -1, 2, 4, \underset{\uparrow}{\pi} \} \longleftrightarrow -z^{3} + 2z^{2} + 4z + \pi \qquad \infty \notin RoC$$

left sided, $n \leq 0$

Example

$$x[n] = \left(-\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{1}{3}\right)^n u[n]$$

$$\left(-\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1+\frac{1}{2}z^{-1}} \qquad |z| > \tfrac{1}{2}$$

$$\left(\tfrac{1}{3}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{3}$$

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{2-\frac{1}{6}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

RoC : $|z| > \frac{1}{2} \cap |z| > \frac{1}{3} = |z| > \frac{1}{2}$

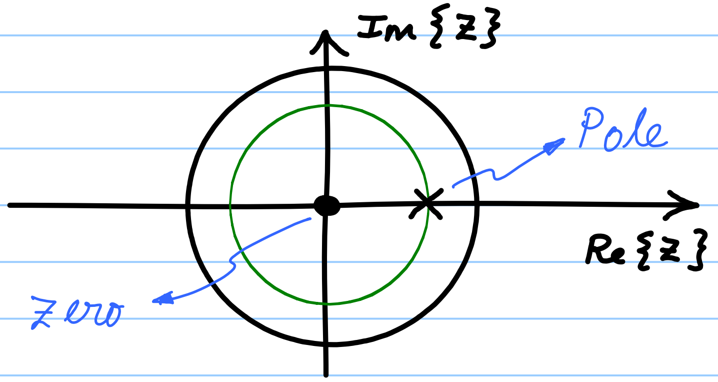
The final RoC is the INTERSECTION of the individual RoCs.

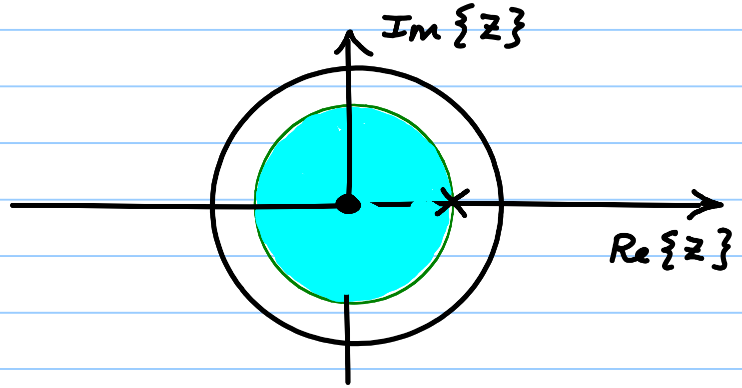
Poles and Zeros

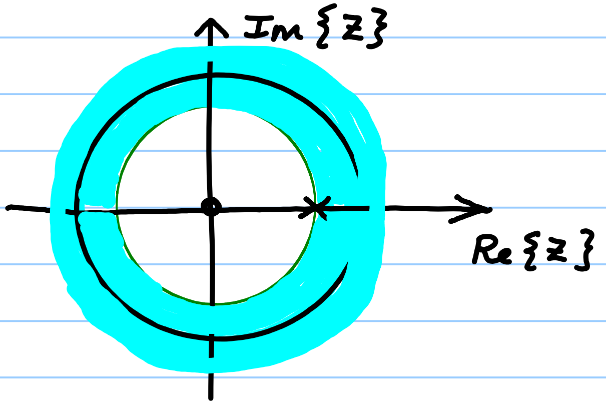
$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Pole: $z=a$

Zero: $z=0$







RoC : $|z| < |a| \Rightarrow x[n] = -a^{n} u[-n-1]$

RoC: $|z| > |a| \Rightarrow x[n] = a^{n} u[n]$

Suppose $e^{jw} \in$ RoC. We can then evaluate $H(z)$ at $z = e^{jw}$

$$H(z)\Big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} h[n]\, e^{-jwn}$$

$$= H(e^{jw})$$ Discrete-Time Fourier Transform

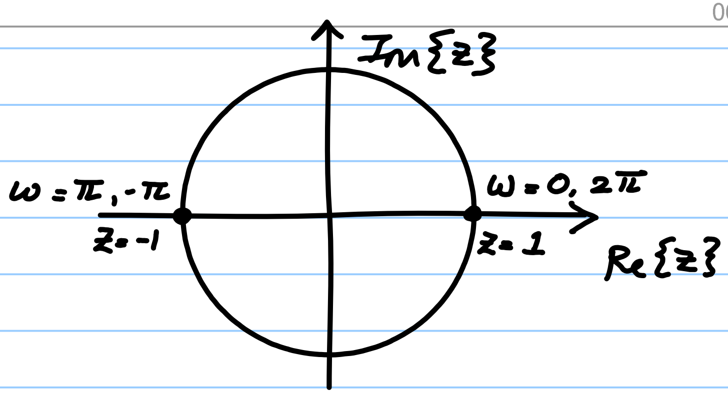
DTFT is a complex function of a single real variable ω. Hence can be plotted in $\boxed{\text{one 3-D plot}}$.

Typically two 2-D plots are shown: mag. vs. ω & phase vs. ω

Note Title 06-08-2013

EE 5330 Aug. 6, 2013

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$



Some books use the notation $X(\omega)$ for the DTFT. In this case,

$$X(\omega)\Big|_{\omega=0} = X(0)$$

OTOH, Watch out for notation tripping you!

$$X(e^{j\omega})\Big|_{\omega=0} = X(1)$$

$X(e^{j\omega})$ makes explicit the 2π-periodicity

Example:

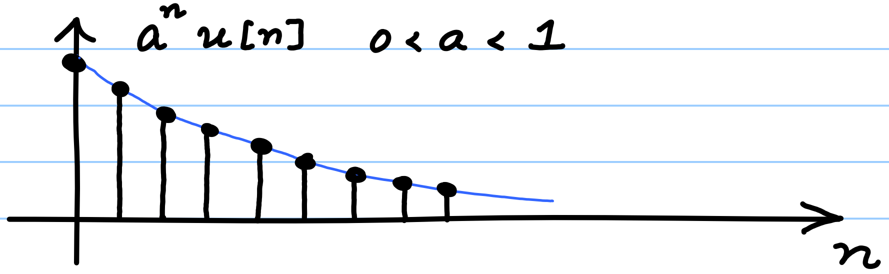
$$x[n] = a^n u[n]$$

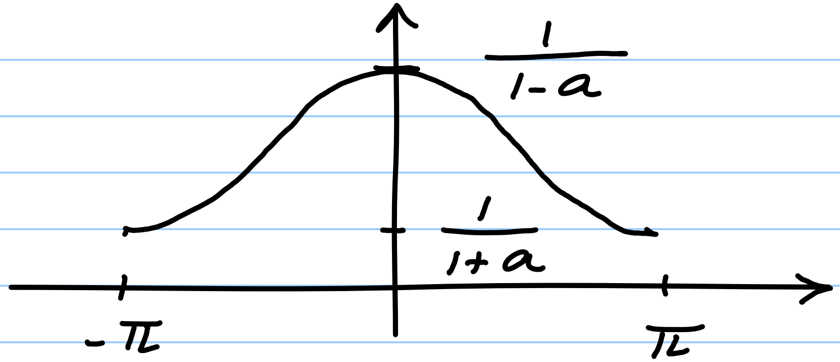
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1-ae^{-j\omega}} \quad |a|<1$$

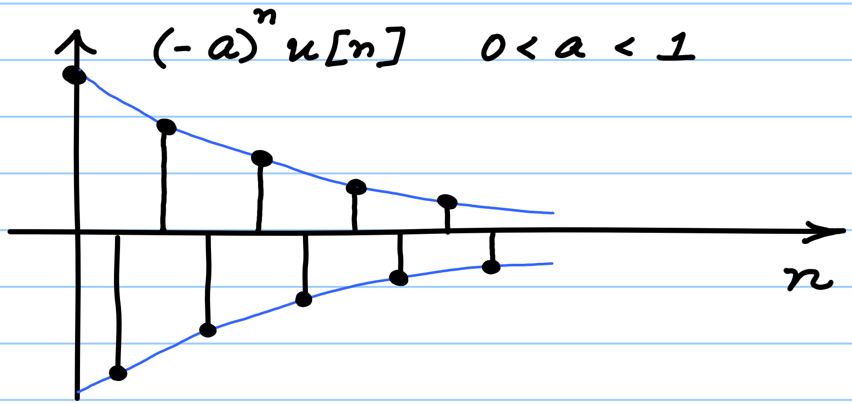
$$|X(e^{j\omega})| = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

If $0 < a < 1$,

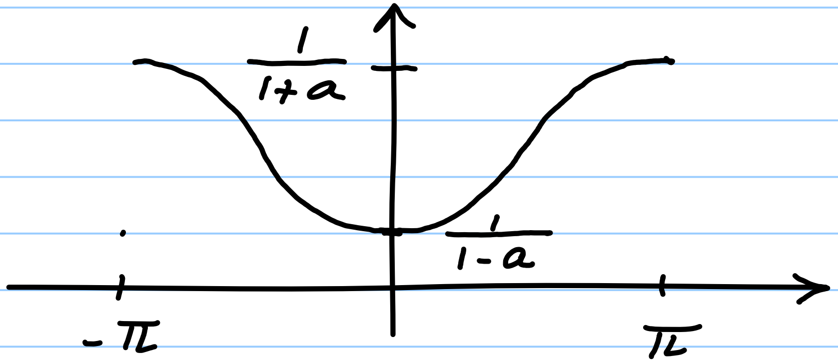




" lowpass signal "



$(-a)^n u[n] \quad 0 < a < 1$



" highpass signal "

Example

$x[n] = 2^n u[n]$

DTFT does not exist!

OTOH,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Let $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

Can think of $X(z)$ as the DTFT of $x[n] r^{-n}$

Hence, for $2^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} 2^n r^{-n} e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{r}\right)^n e^{-jwn}$$

$$= \frac{1}{1 - \frac{2}{r} e^{-jw}}$$ provided $r > 2$

$$= \frac{1}{1 - 2z^{-1}}$$ $|z| > 2$

since $z = re^{jw}$

Note that if $x[n] = a^n$ (not $a^n u[n]$)

then DTFT does not exist even if $|a| < 1$.

This is because the summation does not converge over the range $-\infty < n < 0$.

We will see later that a^n does not possess Z-transform either.

If $x[n] = 1$ (i.e. a^n with $a = 1$), DTFT exists

$$1 \longleftrightarrow \pi\,\delta(\omega) + \frac{1}{1 - e^{-j\omega}} \qquad -\pi \leq \omega < \pi$$

impulse shows up! $\longrightarrow$

(we will derive this later)

Convergence of the Z-transform

(1) Uniform Convergence.

Consider the unilateral z-transform defined as

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

X(z) is said to converge uniformly at $z = z_0$ to $X(z_0)$ if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \quad s.t. \ \forall m > N \quad \left| \sum_{n=0}^{m} x[n] z_0^{-n} - X(z_0) \right| < \varepsilon$$

In this course, we will focus only on absolute convergence, discussed next.

(2) Absolute Convergence:

Let $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X(z)$ converges absolutely at $z = z_0$ to $X(z_0)$ if $|X(z_0)|$ is finite

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

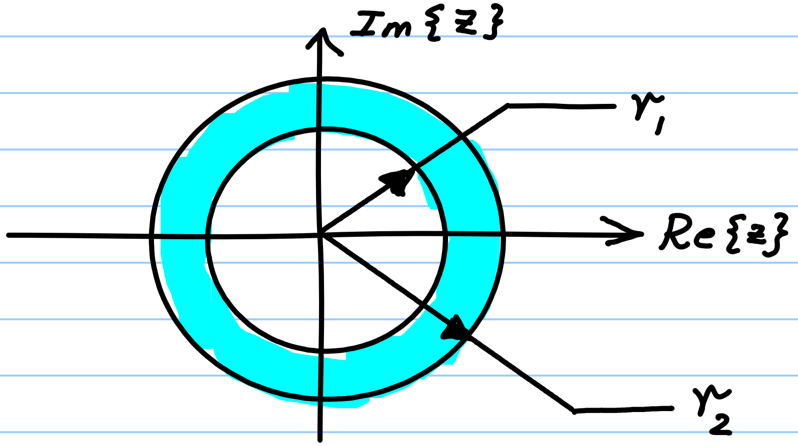
$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} \right| \quad \text{since } z = re^{j\omega}$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| r^{-n}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} = \underbrace{\sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}}_{\text{causal part}} + \underbrace{\sum_{n=1}^{\infty} |x[-n]|\, r^{n}}_{\text{anti-causal part}}$$

If $\exists\, r_1$ s.t. the above converges, then the sum converges for all $r > r_1$. This is because $\frac{1}{r^n} < \frac{1}{r_1^n}$ for $r > r_1$

If $\exists\, r_2$ s.t. the above converges, then the sum converges for all $r < r_2$. This is because $r^n < r_1^n$ for $r < r_2$



Thus, the convergence region is, in general, an ANNULAR region of the form $r_1 < |z| < r_2$

r_1 can be as small as zero
r_2 can be as large as infinity

If $z_0 = r_0 e^{j\omega_0} \in RoC$, then $|z_0| \in RoC$, i.e. if it converges at $\omega = \omega_0$, it converges for all $\omega \in [0, 2\pi) \Rightarrow r_0 e^{j\omega} \in RoC$

Example

$$a^n u[n] \longleftrightarrow \frac{1}{1 - a z^{-1}} \qquad |z| > |a| \Rightarrow r_1 = |a|$$

$$r_2 = \infty$$

$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1 - a z^{-1}} \qquad |z| < |a| \Rightarrow r_1 = 0$$

$$r_2 = |a|$$

Example

Let $x[n] = \frac{1}{n} \qquad n = 1, 2, \ldots.$

What can you say about the RoC ?

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Does this series converge ?

Compare the following two series :

$$1, \underbrace{\frac{1}{2}}_{1 \text{ term}}, \underbrace{\frac{1}{3}, \frac{1}{4}}_{2 \text{ terms}}, \underbrace{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}}_{4 \text{ terms}}, \underbrace{\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \ldots, \frac{1}{16}}_{8 \text{ terms}}, \frac{1}{17}, \ldots$$

$$1, \quad \underbrace{\frac{1}{2}}_{\substack{\text{sum} \\ \frac{1}{2}}}, \quad \underbrace{\frac{1}{4}, \quad \frac{1}{4}}_{\substack{\text{sum} \\ \frac{1}{2}}}, \quad \underbrace{\frac{1}{8}, \quad \frac{1}{8}, \quad \frac{1}{8}, \quad \frac{1}{8}}_{\substack{\text{sum} \\ \frac{1}{2}}}, \quad \underbrace{\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \quad \ldots\ldots, \quad \frac{1}{16}}_{\substack{\text{sum} \\ \frac{1}{2}}}, \frac{1}{32}, \ldots$$

Term-by-term, the 1st series is greater than the 2nd series, and the second series diverges. By the comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \infty$$

Note: If the series converges, $a_n \rightarrow 0$
If $a_n \rightarrow 0$, the series need not converge.

Now, what can you say about RoC of $X(z)$? Specifically, is the unit circle part of the RoC?

EE 5330 Aug. 7, 2013

$x[n] = \frac{1}{n} \quad n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty \Rightarrow e^{j\omega} \notin$ RoC

RoC : $|z| > 1$

Example

$x[n] = \frac{1}{n^2} \quad n = 1, 2, \ldots$

$e^{j\omega} \in$ RoC can show RoC : $|z| \geq 1$

The point that we wish to illustrate with the above example is that the inequality need not always be strict.

However, for the major class of z-transforms that we encounter in this course, the inequality will be strict.

The z-transform is analytic in the region of convergence. An analytic function satisfies the

Cauchy-Riemann equations. That is, if

$$f(x+jy) = u(x,y) + j\,v(x,y)$$

is analytic, then

$$u_x = v_y$$

$$u_y = -v_x$$

where $u_x \equiv \frac{\partial}{\partial x} u(x,y)$ and so on.

Satisfying the CR equations alone does not guarantee analyticity. What is needed is the following.

Looman - Menchoff Theorem

Let $f = u + jv$ be defined on a domain D such that

(i) f is continuous on D

(ii) u_x, u_y, v_x, v_y exist everywhere on D (but not necessarily continuous)

(iii) u and v satisfy the CR equations.

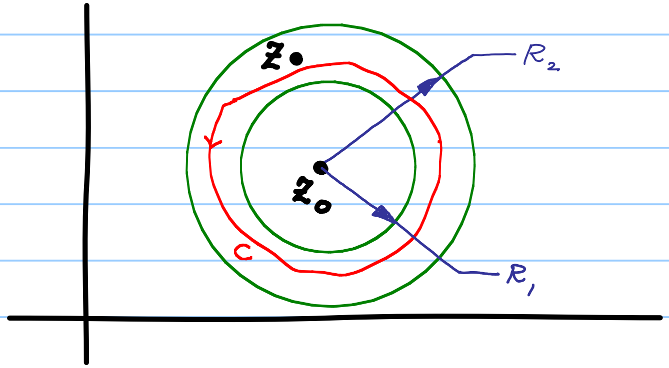
Then f is holomorphic on D. The term analytic is also used interchangeably

The z-transform $X(z)$ of $x[n]$ is analytic in the RoC.
The RoC cannot contain singularities.

The z-transform is in the form of a Laurent Series, whose definition is given below:

Laurent Series

Suppose f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$ centred at z_0. C is as shown in the figure. Then, at each point in the domain, $f(z)$ has the representation



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$R_1 < |z| < R_2$$

where

$$a_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \qquad n = 1, 2, 3, \ldots$$

The above can be combined into a single expression:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad R_1 < |z-z_0| < R_2$$

where

$$c_n = \frac{1}{2\pi j} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad n = 0, \pm 1, \pm 2, \ldots$$

Examples

$$X(z) = e^{z}$$

$$= \sum_{n=0}^{\infty} \frac{z^{n}}{n!} \qquad |z| < \infty$$

$$X(z) = e^{1/z}$$

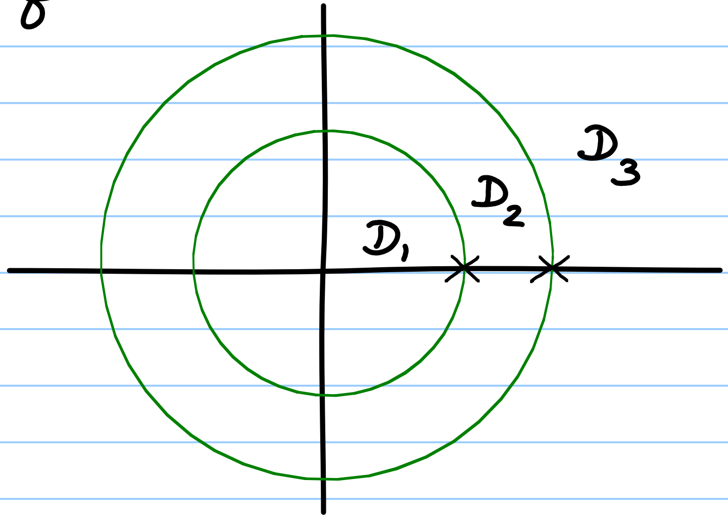
$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \qquad 0 < |z| < \infty$$

Example

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

The singularities in $X(z)$ are at $z=1$, $z=2$.

Consider the series expansion of $X(z)$ in 3 different regions.



D_1 : $|z| < 1$

D_2 : $1 < |z| < 2$

D_3 : $2 < |z| < \infty$

In D_1, i.e., $|z| < 1$, $|z/2| < 1$

$$X_1(z) = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-z/2} \qquad |z| < 1$$

$$= -\sum_{n=0}^{\infty} z^{n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n} \qquad |z| < 1$$

$$= \sum_{n=0}^{\infty}\left(2^{-n-1} - 1\right) z^{n} \qquad |z| < 1$$

The above expansion contains only +ve powers of z.

In D_2, i.e., $1 < |z| < 2$, $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{z}{2}\right| < 1$. Hence,

$$X_2(z) = \frac{1}{z}\,\frac{1}{1-\,1/z} + \frac{1/2}{1-\,z/2} \qquad 1 < |z| < 2$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty}\frac{z^{n}}{2^{n+1}} \qquad 1 < |z| < 2$$

The above expansion contains both +ve and -ve powers of z.

In D_3, i.e., $2 < |z| < \infty$, $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{2}{z}\right| < 1$. Hence,

$$X_2(z) = \frac{1}{z} \frac{1}{1 - 1/z} + \frac{1}{z} \frac{1}{1 - \frac{2}{z}} \qquad 2 < |z| < \infty$$

$$= \sum_{n=1}^{\infty} z^{-n} + \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} \qquad 2 < |z| < \infty$$

$$= \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n} \qquad 2 < |z| < \infty$$

The above expansion contains only -ve powers of z.

Thus $X(z)$ has 3 different series expansions in the 3 different regions. Each series expansion is valid only in one particular region.

$X_1(z) = X(z)$ in D_1 i.e., $X_1(z) = X(z)\Big|_{D_1}$

$X_2(z) = X(z)$ in D_2 i.e., $X_2(z) = X(z)\Big|_{D_2}$

$X_3(z) = X(z)$ in D_3 i.e., $X_3(z) = X(z)\Big|_{D_3}$

Isolated Singular Point:

z_0 is a singular point of $X(z)$ if it fails to be analytic at z_0. A singular point is isolated if, in addition, there is a deleted neighbourhood $0 < |z-z_0| < \epsilon$ of z_0 throughout which X is analytic.

Example

$$X(z) = \frac{z+1}{z^3(z^2+1)}$$

Singularities are at $z = 0, \pm j$

Example

$$X(z) = \frac{1}{\sin(\pi/z)}$$

Singular points: $z = 0$ and $z = \frac{1}{n}$ $n = \pm 1, \pm 2, \ldots$

all lying on the real axis from $z = -1$ to $z = 1$.

Singularities at $z = \frac{1}{n}$ are isolated.

Singularity at $z = 0$ is not isolated because any ε-neighbourhood around $z = 0$ will contain other singularities.

Three Types of Singularities

Recall

$$X(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{"principal part"}}$$

where the expansion is in a punctured disk $0 < |z-z_0| < R$

If $b_m \neq 0$ but $b_{m+1} = b_{m+2} = \cdots = 0$, then the singularity at $z = z_0$ is a pole of order m.

Example

$$X(z) = \frac{z^2 - 2z + 3}{z - 2} = z + \frac{3}{z - 2}$$

$$= 2 + (z-2) + \frac{3}{z-2} \qquad 0 < |z-2| < \infty$$

Hence we conclude that there is pole of order 1 at $z = 2$.

Example

$$X(z) = \frac{1}{z^2(1+z)}$$

$$= \frac{1}{z^2}\left(1 - z + z^2 - z^3 + \cdots\right) \qquad 0 < |z| < 1$$

$$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots \qquad 0 < |z| < 1$$

Hence $X(z)$ has a pole of order 2 at $z = 0$.

In the above two examples, the order and location of the poles can be inferred from the expression for $X(z)$ directly, without going through the series expansion.

This is because $X(z)$ was a rational function, i.e., a ratio of polynomials in z, of the form $P(z)/Q(z)$

The power of the series expansion method is made clear in the following example.

Example

$$X(z) = \frac{\sinh z}{z^4}$$

$$= \frac{1}{z^4}\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \right] \qquad 0 < |z| < \infty$$

$$= \frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots \qquad 0 < |z| < \infty$$

There is a pole of order 3 at $z = 0$.

Example

$$X(z) = \frac{1 - \cos z}{z^2}$$

$$= \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] \qquad 0 < |z| < \infty$$

$$= \frac{1}{2!} + \frac{z^2}{2!} - \frac{z^2}{4!} + \frac{z^4}{6} - \cdots \qquad 0 < |z| < \infty$$

If we define $X(0) = \frac{1}{2}$, then $X(z)$ has no singularities.

Thus, $X(z) = \frac{1 - \cos z}{z^2}$ has a removable singularity.

Removable Singularity

If $X(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ over $0 < |z-z_0| < R$, then $X(z)$ has a removable singularity at $z = z_0$. If we now define $X(z_0) = a_0$, $X(z)$ becomes ENTIRE, i.e., analytic over the entire z-plane.

Essential Singularity

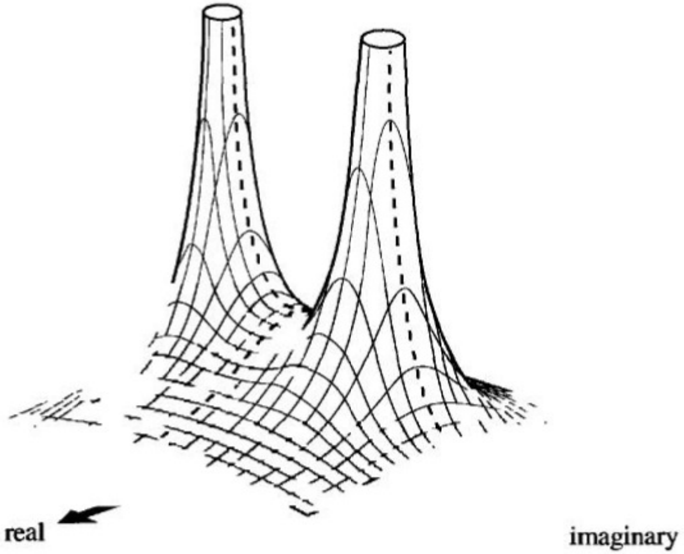
If the series representation of $X(z)$ over the punctured disk $0 < |z - z_0| < R$ contains all negative powers of $z - z_0$, then $z = z_0$ is an essential singularity.

Example

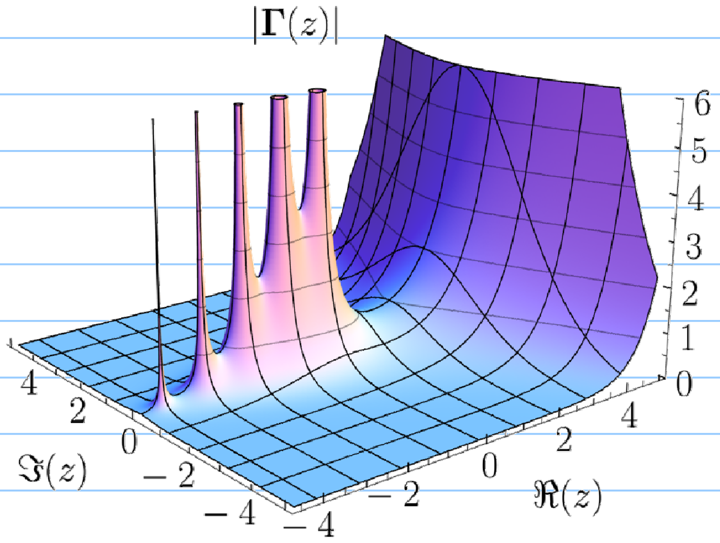
$$X(z) = e^{1/z}$$

$$= 1 + \frac{1}{1!\,z} + \frac{1}{2!\,z^2} + \cdots \qquad 0 < |z| < \infty$$

$z = 0$ is an essential singularity.



From *Visual Complex Analysis* by T. Needham, Oxford University Press, 1999, p. 66.



The absolute value of the Gamma function. This shows the function becomes infinite at the poles at n = -1, -2, -3, ... (Wikipedia, "Pole (complex analysis)")

An isolated singular point z_0 of a function $X(z)$ is a pole of order m if and only if $X(z)$ can be written in the form

$$X(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic and non-zero at z_0

EE 5330 Aug. 12, 2013

Properties of RoC for rational $X(z)$

1) RoC is, in general, an annular region of the form $r_1 < |z| < r_2$ [r_1 can be as small as 0, r_2 can be as large as ∞]

2) If $e^{j\omega} \in$ RoC, then the DTFT can be obtained by replacing z by $e^{j\omega}$

3) RoC cannot contain poles

4) If $x[n]$ is a Finite duration signal, then the RoC is the entire z-plane, except possibly 0 and/or ∞

5) If $x[n]$ is a right-sided sequence, then the RoC is outside of a certain circle. ∞ may or may not belong to the RoC

6) If $x[n]$ is a left-sided sequence, then the RoC is inside of a certain circle. 0 may or may not belong to the RoC

7) If $x[n]$ is a two-sided infinite sequence, then the RoC is in between two circles.

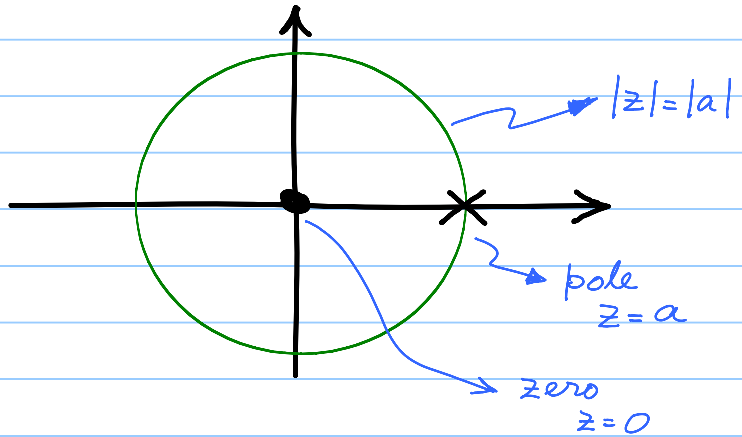
8) RoC must be a connected region. If the region is disconnected, the series expansion fails since it can be valid in only one region ⇒ fails to be valid in the other regions.

Poles and Zeros Revisited

$$H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Pole : $z = a$

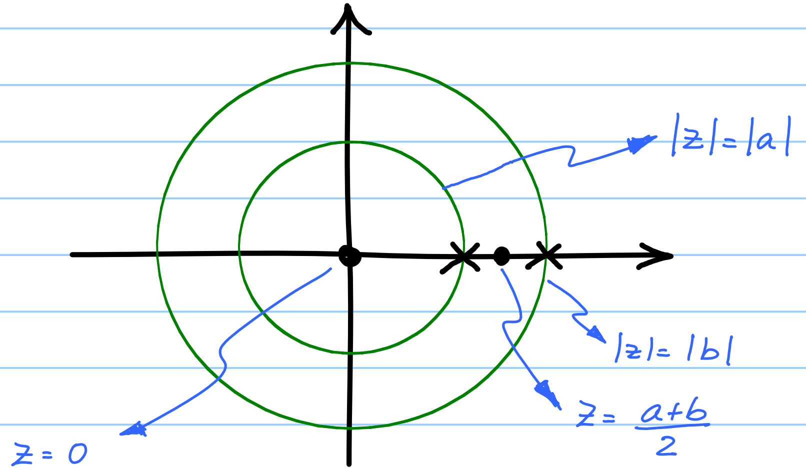
Zero : $z = 0$



$$a^n u[n] + b^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

$$= \frac{z\left(z-\frac{a+b}{2}\right)}{(z-a)(z-b)}$$

RoC : $|z|>|a| \cap |z|>|b|$



$H(z) = \dfrac{1}{z-a}$

Pole : $z = a$

Zero : ?

To investigate behaviour at $z = \infty$, make the transformation $z = \frac{1}{s}$

$$Y(s) = X(z)\Big|_{z=\frac{1}{s}}$$

$$= \frac{1}{\frac{1}{s} - a}$$

$$= \frac{s}{1-as}$$

$\Rightarrow$ $s = 0$ is a zero of $Y(s)$

$\Rightarrow$ $z = \infty$ is a zero of $X(z)$

$$\{b_0, b_1, b_2, \ldots, b_M\} \longleftrightarrow \overbrace{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}^{B(z)}$$

RoC : $|z| > 0$

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{z^M} = \frac{B_1(z)}{z^M}$$

Zeros : roots of $B_1(z)$ There are M zeros in the finite z-plane

Poles : M^{th} order pole at $z=0$

A pole or a zero at $z=0$ is called a TRIVIAL pole or zero

Neglecting the trivial pole at $z=0$, the above is called an "All-Zero Filter". This Filter is FIR.

Similarly,

$$H(z) = \frac{1}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

is called an "All-Pole Filter". It has a trivial zero of order N. This Filter is IIR.

In general, $H(z) = \frac{B(z)}{A(z)}$ is called as a "Pole-Zero Filter"

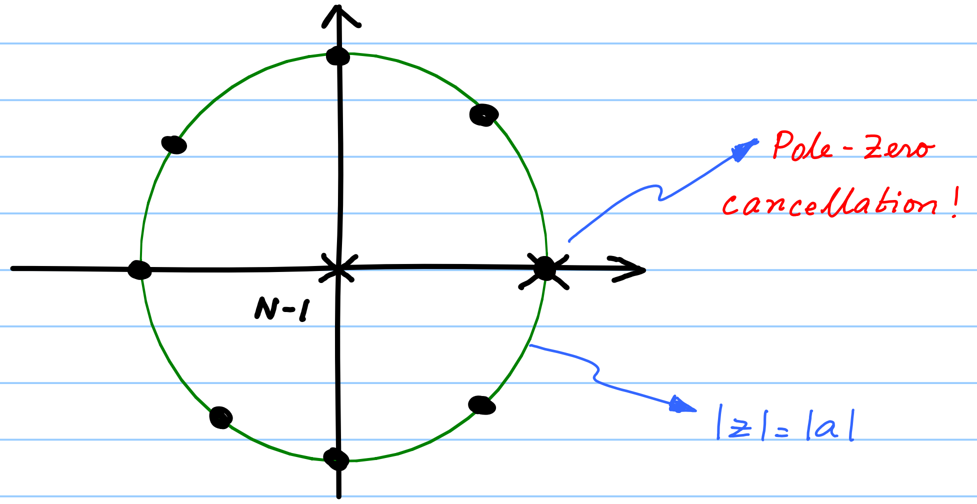
If there are uncancelled non-trivial poles, this filter will be IIR

Let $h[n] = a^n \qquad 0 \leq n \leq N-1$

$$H(z) = 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-(N-1)} \qquad |z| > 0$$

$$= \frac{1 - a^N z^{-N}}{1 - a z^{-1}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$



N zeros lie on the circle $|z| = |a|$. The pole at $z = a$ cancels with the zero at $z = a$. There is an $(N-1)^{th}$ order trivial pole.

In the time-domain, the input-output relationship can be shown to take up either of the following forms:

$$y[n] = x[n] + a\,x[n-1] + \cdots + a^{N-1}x[n-N+1]$$

or

$$y[n] = a\,y[n-1] + x[n] - a^{N}x[n-N]$$ ← Corresponds to a recursive implementation of the above non-recursive difference eqn.

Any pole introduced in the recursive implementation must necessarily get cancelled, since the given filter is FIR.

FIR filters cannot have uncancelled non-trivial poles!

Properties of the Z-Transform:

1) Linearity

$$y[n] = a_1 x_1[n] + a_2 x_2[n] \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

$$RoC_y \supseteq RoC_{x_1} \cap RoC_{x_2}$$

The RoC is atleast as large as the intersection of the two RoCs, but can be larger if there are some pole-zero cancellations.

$$y[n] = a_1 x_1[n] + a_2 x_2[n] \overset{DTFT}{\longleftrightarrow} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

$$x_1[n] = a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$x_2[n] = a^n u[n-N] \longleftrightarrow \frac{a^N z^{-N}}{1-az^{-1}} \qquad |z| > |a|$$ (See next property for derivation)

$$x_1[n] - x_2[n] \longleftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}} = 1 + az^{-1} + \cdots + a^{N-1} z^{-(N-1)}$$

The RoC is $|z|>0$ and larger than $|z|>|a|$ because the pole at $z=a$ gets cancelled.

2) Time Shift

$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$$ RoC is identical except possibly for the addition or deletion of 0 and/or ∞

$$y[n] = x[n-n_0] \xleftrightarrow{DTFT} Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Note that $|Y(e^{j\omega})| = |X(e^{j\omega})|$

$x[n] = 1 \quad -N \leq n \leq N$

$$X(z) = z^N + z^{N-1} + \cdots + z + 1 + z^{-1} + \cdots + z^{-N}$$

$$= \frac{z^N\left(1 - z^{-\overline{2N+1}}\right)}{1 - z^{-1}} \qquad 0 < |z| < \infty$$

$$= \frac{z^N - z^{-N-1}}{1 - z^{-1}} = \frac{z^{N+\frac{1}{2}} - z^{-N-\frac{1}{2}}}{z^{1/2} - z^{-1/2}}$$

$$y[n] = x[n-N] \Longrightarrow Y(z) = z^{-N} X(z)$$

$$Y(z) = \frac{1 - z^{-(2N+1)}}{1 - z^{-1}} \qquad |z| > 0$$

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$

$$= \frac{e^{j\omega(N+1/2)} - e^{-j\omega(N+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin\left[(2N+1)\,\omega/2\right]}{\sin(\omega/2)}$$

Dirichlet kernel

[See the command 'diric' in MATLAB]

$$Y(e^{j\omega}) = e^{-j\omega N}\,\frac{\sin\left[(2N+1)\,\omega/2\right]}{\sin(\omega/2)}$$

Transform of $a^{n}u[n-N]$ can be easily obtained using the delay property $\Big\}$

$$a^{N}\,\underbrace{a^{n-N}u[n-N]}_{x[n-N]} \longleftrightarrow a^{N}\,\frac{z^{-N}}{1-az^{-1}}$$

3) Exponential Multiplication

$$\gamma^{n}\, x[n] \longleftrightarrow X(z/\gamma) \qquad |z/\gamma| \in RoC_{x}$$

$$u[n] \longleftrightarrow \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$a^{n}\, u[n] \longleftrightarrow \frac{1}{1-(z/a)^{-1}} \qquad |z/a| > 1$$

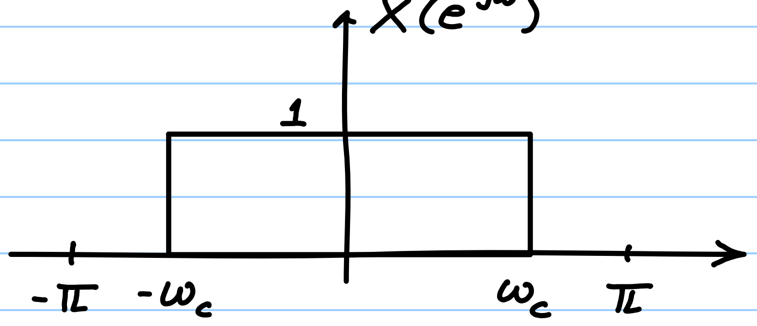
$$= \frac{1}{1-az^{-1}} \qquad |z| > |a| \quad \text{as before}$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(z/e^{j\omega_0})$$

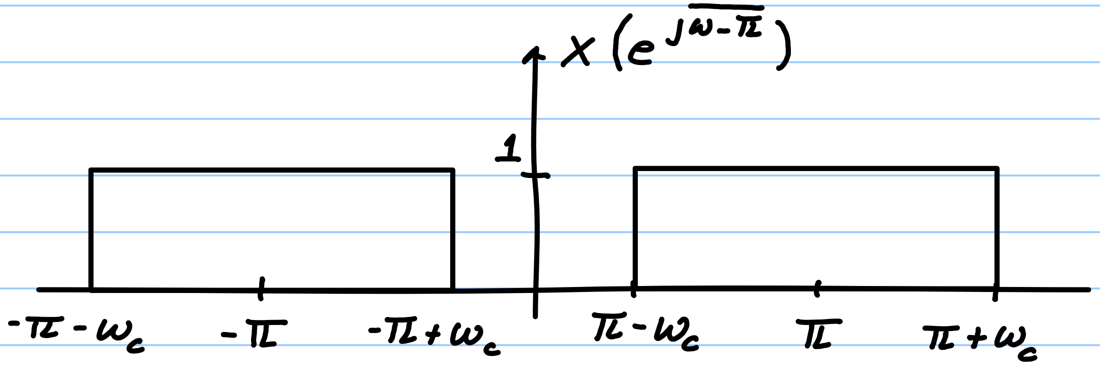
$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j\omega}/e^{j\omega_0}) = X(e^{j\overline{\omega-\omega_0}})$$ Modulation property !

$$(-1)^n x[n] \xleftrightarrow{DTFT} X(e^{j\overline{\omega-\pi}}) = X(e^{j\overline{\omega+\pi}})$$

$$(-1)^n x[n] \xleftrightarrow{Z} X(-z)$$

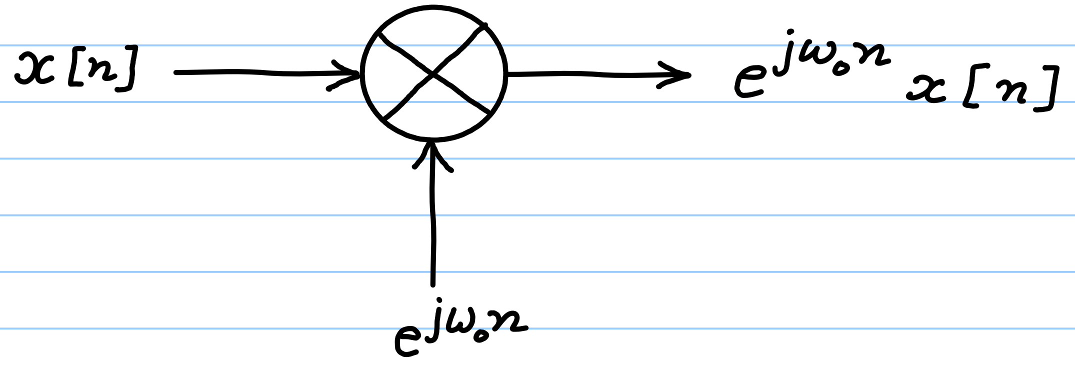


lowpass



highpass

Modulator block diagram:



Is the modulator a linear system ?

Is it time-invariant ?

Using the exponential multiplication property, derive the following:

$$r^n \cos\omega_o n \; u[n] \xleftrightarrow{\;z\;} \frac{1 - r\cos\omega_o z^{-1}}{1 - 2r\cos\omega_o z^{-1} + r^2 z^{-2}} \qquad |z| > r$$

Plot the poles and zeros

Hint: Replace $\cos\omega_o n$ by $\dfrac{e^{j\omega_o n} + e^{-j\omega_o n}}{2}$

EE5330 Aug. 19, 2013

Recall the exponential multiplication property:

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$\gamma^n x[n] \longleftrightarrow X(z/\gamma) \qquad |\gamma| r_1 < |z| < |\gamma| r_2$$

Let $y[n] = \gamma^n x[n]$

$$\Rightarrow \quad Y(z) = X(z/\gamma)$$

$$\Rightarrow \quad Y(\gamma z) = X(z)$$

Suppose $X(z) = \dfrac{P(z)}{Q(z)}$

$$Y(z) = \frac{P(z/\gamma)}{Q(z/\gamma)}$$

If z_0 is a zero of $X(z)$, i.e. $X(z_0) = 0 \Rightarrow P(z_0) = 0$

then $Y(\gamma z_0) = \frac{P(z_0)}{Q(z_0)} = 0 \Rightarrow \gamma z_0$ is a zero of $Y(z)$

Similarly, if z_1 is a pole of $X(z)$, i.e., $Q(z_1) = 0$

then $Y(\gamma z_1) = \frac{P(z_1)}{Q(z_1)} \rightarrow \infty \Rightarrow \gamma z_1$ is a pole of $Y(z)$

All poles and zeros get multiplied by γ

Geometrically, each pole/zero gets scaled by $|\gamma|$ and rotated by $\measuredangle \gamma$.

4) Differentiation in the z-domain

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$? \longleftrightarrow -z\frac{dX}{dz} \qquad \text{RoC } ?$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}z^{-n}$$

This operation is allowed because the power series is absolutely convergent in the RoC

$$= \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1}$$

$$-z\frac{dX}{dz} = \sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}$$

Hence, $n\,x[n] \longleftrightarrow -z\frac{dX(z)}{dz}$

Since $X(z)$ is analytic in the RoC, it can be differentiated infinite no. of times. Hence, the above property can be repeatedly applied

The RoC of $-z\frac{dX}{dz}$ is the same as the RoC of $X(z)$ except possibly for the deletion of the boundary circle (if it were part of the original RoC)

Example

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-z\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{(-z)(-1)(-a)(-z^{-2})}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

i.e., $$n a^n u[n] \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1) a^{n+1} u[n+1] \longleftrightarrow \frac{a}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$(n+1)\,a^{n}\,u[n+1] \longleftrightarrow \frac{1}{(1-az^{-1})^{2}} \qquad |z|>|a|$$

Can be rewritten as,

$$(n+1)\,a^{n}\,u[n] \longleftrightarrow \frac{1}{(1-az^{-1})^{2}} \qquad |z|>|a|$$

Repeat the above steps by starting with $\frac{1}{1-az^{-1}}$ but with RoC $|z|<|a|$. At what index does the first non-zero sample begin?

5) Complex Conjugation

$$x^*[n] \longleftrightarrow X^*(z^*) \qquad r_1 < |z| < r_2$$

$$\sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = \left[\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right]^*$$

$$= X^*(z^*) \qquad r_1 < |z| < r_2$$

The corresponding property for the DTFT is :

$$X^*(z^*) \Big|_{z=e^{j\omega}} = X^*(e^{-j\omega})$$

If $x[n] \in \mathbb{R}$, then $x^*[n] = x[n]$

Hence, for real-valued sequences, the z-transform satisfies

$$X(z) = X^*(z^*)$$

For such sequences, if z_0 is a zero of $X(z)$, then $X(z_0) = 0$.

Therefore, $X(z_0) = 0$

$$\Rightarrow X(z_0) = X^*(z_0^*)$$

$$\Rightarrow X^*(z_0^*) = 0$$

$$\Rightarrow X(z_0^*) = 0$$

$\Rightarrow$ z_0^* is also a zero of $X(z)$

Thus, zeros occur in complex conjugate pairs.
Similarly, it is easy to see that poles also occur in complex conjugate pairs.

Also, for real-valued sequences, $X(e^{j\omega}) = X^*(e^{-j\omega})$ [conjugate even]

$$\Rightarrow |X(e^{j\omega})| = |X^*(e^{-j\omega})|$$

$$= |X(e^{j\omega})|$$ DTFT mag. is an even function of ω

Exercise

Starting from $X(e^{j\omega}) = X^*(e^{-j\omega})$, show that

$\measuredangle X(e^{j\omega})$ is an odd function of ω

6) Time Reversal

$$x[n] \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$x[-n] \longleftrightarrow X(z^{-1}) \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

This operation makes a causal sequence non-causal and vice-versa

Example

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

Using the time-reversal property,

$$a^{-n} u[-n] \longleftrightarrow \frac{1}{1-az} \qquad |z| < \frac{1}{|a|}$$

$$\frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$a^{-n}u[-n] \longleftrightarrow \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$a^{-n-1}u[-n-1] \longleftrightarrow \frac{-a^{-1}}{1-a^{-1}z^{-1}} \qquad |z| < \frac{1}{|a|}$$

$$-a^{-n}u[-n-1] \longleftrightarrow \frac{1}{1-a^{-1}z^{-1}} \qquad |z| < |a^{-1}|$$

Let $b = a^{-1}$. Hence,

$$-b^{n}u[-n-1] \longleftrightarrow \frac{1}{1-bz^{-1}} \qquad |z| < |b|$$

as before.

EE 5330 Aug. 22, 2013

We can now see why $x[n] = 1$ has no z-transform.

Recall

$$u[n] = \frac{1}{1-z^{-1}} \qquad |z| > 1$$

Hence,

$$u[-n] = \frac{1}{1-z} \qquad |z| < 1$$

$$u[-n-1] = \frac{z}{1-z}$$

$$= \frac{-1}{1-z^{-1}} \qquad |z| < 1$$

$$\begin{tikzcd} x[n] = 1 = & u[n] \arrow[d, "\text{RoC}"] & + & u[-n-1] \arrow[d, "\text{RoC}"] \\ & \lvert z \rvert > 1 & & \lvert z \rvert < 1 \end{tikzcd}$$

Since $|z| > 1 \cap |z| < 1 = \phi \Rightarrow x[n] = 1$ has no z-transform!

Similarly, a^n has no z-transform

Exercise

Find the z-transform of $a^{|n|}$. Plot the pole-zero plot.
For what values of 'a' does the transform exist?

Hint: $a^{|n|} = a^n u[n] + a^{-n} u[-n-1]$

Observe the differences between modulation and time-reversal.

$$(-1)^n x[n] \longleftrightarrow X(-z)$$

$$x[-n] \longleftrightarrow X(z^{-1})$$

$$(-1)^n x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\overline{\omega \pm \pi}})$$

$$x[-n] \overset{DTFT}{\longleftrightarrow} X(e^{-j\omega})$$

If the $X(\omega)$ notation is used, this will be written as $X(-\omega)$. Do not confuse this with $X(-z)$. Be careful when comparing books that use different notation

7) Time-Domain Convolution

Let $p[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$

Then $P(z) = X(z)Y(z)$ $\quad RoC \supseteq RoC_x \cap RoC_y$

Proof:

$$P(z) = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x[k]\,y[n-k]\right] z^{-n}$$

$$\overset{?}{=} \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}\, Y(z)$$

$$= X(z)\,Y(z)$$

Hence $P(z) = X(z)Y(z)$

For the DTFT,

$$x[n] * y[n] \xleftrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$$

This property forms the basis for

$$\boxed{\text{FREQUENCY SELECTIVE FILTERING}}$$

Example

$$x[n] = a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$y[n] = -b^n u[-n-1] \longleftrightarrow \frac{1}{1-bz^{-1}} \qquad |z| < |b|$$

$$x[n] * y[n] \longleftrightarrow \frac{1}{(1-az^{-1})(1-bz^{-1})} \qquad |a| < |z| < |b|$$

Exercise

Evaluate the convolution in the time domain — make sure you get the limits fixed correctly for $n<0$ and $n \geqslant 0$.

What happens when $a \rightarrow b$?

Repeat for $a^n u[n] * b^n u[n]$

8) Product Theorem

$$x[n]\, y[n] \longleftrightarrow \frac{1}{2\pi j} \oint_C X(\xi)\, Y(z/\xi)\, \frac{d\xi}{\xi}$$

Knowledge of inversion integral is needed to prove this.

9) Initial Value Theorem :

Let $x[n] = 0$ for $n < 0$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$\operatorname{Lt}_{z \to \infty} X(z) = x[0]$$

If $x[n] = 0$ for $n < 1$, then

$$X(z) = x[1]z^{-1} + x[2]z^{-2} + \cdots$$

In this case,

$$\operatorname{Lt}_{z \to \infty} z\,X(z) = x[1]$$ and so on.

10) Final Value Theorem

Let $x[n] = 0$ for $n < M$

Define $v[n] = x[n] - x[n-1]$

Hence $V(z) = (1-z^{-1})X(z)$

$$V(z) = \sum_{n=M}^{\infty} (x[n] - x[n-1]) z^{-n}$$

$$\underset{z\to 1}{Lt}\ V(z) = \underset{z\to 1}{Lt} \sum_{n=M}^{\infty} (x[n] - x[n-1]) z^{-n}$$

$$\overset{?}{=} \sum_{n=M}^{\infty} (x[n] - x[n-1])$$

$$= \underset{N\to\infty}{Lt} \sum_{n=M}^{N} \left(x[n] - x[n-1] \right)$$

$$= \underset{N\to\infty}{Lt} \left[\begin{array}{l} \cancel{x[M]} - \cancelto{0}{x[M-1]} \\ + \cancel{x[M+1]} - \cancel{x[M]} \\ + x[M+2] - \cancel{x[M+1]} \\ + \quad \ldots \\ + \cancel{x[N-1]} + x[N-2] \\ + x[N] - \cancel{x[N-1]} \end{array} \right]$$

$$= \underset{N\to\infty}{Lt} \; x[N]$$

$$= x[\infty]$$

Hence, $\underset{z\to 1}{Lt} V(z) = \boxed{\underset{z\to 1}{Lt} (1-z^{-1}) X(z) = x[\infty]}$

Note Title

EE 5330 Aug. 26, 2013

26-08-2013

An alternative version of the Final Value Theorem:

Let the discrete-time signal $x[n]$ have the one-sided z-transform $X_+(z)$ defined as $\sum_{n=0}^{\infty} x[n] z^{-n}$. Then, if $\operatorname{Lt}_{n \to \infty} x[n]$ exists,

$$\operatorname{Lt}_{z \to 1} (z-1) X_+(z) = \operatorname{Lt}_{n \to \infty} x[n]$$

Another variant: For a causal $x[n]$ s.t. $(z-1)X(z)$ can be analytically extended to $\{z : |z| > R\}$ with $R < 1$,

$$\operatorname{Lt}_{n \to \infty} x[n] = \operatorname{Lt}_{z \to 1} (z-1) X(z)$$

Example

$$x[n] = u[n] \longleftrightarrow \frac{1}{1-z^{-1}} \qquad |z| > 1$$

$$= \frac{z}{z-1}$$

Hence

$$\underset{z \to 1}{Lt} (z-1) \frac{z}{z-1} = 1 = x[\infty]$$

Note, however, that for $x[n] = (-1)^n u[n]$, $\underset{z \to 1}{Lt} (z-1) X(z) = 0$ which does not equal $x[\infty]$, as the latter limit does not exist.

11) Parseval's Theorem

Let $x[n] \longleftrightarrow X(z) \quad r_1^x < |z| < r_2^x$

$y[n] \longleftrightarrow Y(z) \quad r_1^y < |z| < r_2^y$

Then,

$$\sum_{n=-\infty}^{\infty} x[n]\, y^*[n] = \frac{1}{2\pi j} \oint_C X(z)\, Y^*(1/z^*)\, \frac{dz}{z}$$

$$r_1^x r_1^y < |z| = 1 < r_2^x r_2^y$$

For the corresponding DTFT property, let $z = e^{j\omega}$

$$dz = j\underbrace{e^{j\omega}}_{z}\,d\omega$$

$$\frac{dz}{z} = j\,d\omega$$

The contour integral now becomes a real-integral over ω; ω varies between $-\pi$ and π

Hence

$$\sum_{n=-\infty}^{\infty} x[n]\,y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,Y^*(e^{j\omega})\,d\omega$$

Exercise

Let $X(e^{j\omega}) = 1$ for $|\omega| < \omega_c$ and zero for $[-\pi, \pi) \setminus [-\omega_c, \omega_c]$

It can be shown that $x[n] = \frac{\sin \omega_c n}{\pi n}$

Using Parseval's Theorem, evaluate

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 \omega_c n}{\pi^2 n^2}$$

Inverse Z-Transform:

First consider the class of $X(z)$ that are rational, i.e., of the form

$$X(z) = \frac{P(z)}{Q(z)}$$

If the input-output relation of a system takes the form of a Linear Constant Coefficient Difference Equation, such as the one given below, then the system transfer function $H(z)$ is a rational one.

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{l=0}^{M} b_l\, x[n-l]$$

Taking z-transform on both sides,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{l=0}^{M} b_l z^{-l}\right]$$

Hence,

$$\frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = H(z) = \frac{B(z)}{A(z)}$$ rational Transfer function

Associated with every LCCDE, there is a rational z-transform.

Conversely, with every rational z-transform, there is an associated LCCDE.

Since an LCCDE can be implemented in practice using multiplier and delay elements, the class of rational TFs is important. This class also models a lot of useful TFs.

Let $$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{\ell=0}^{M} p_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} q_k z^{-k}}$$

$$= z^{N-M} \frac{\sum_{\ell=0}^{M} p_\ell z^{M-\ell}}{\sum_{k=0}^{N} q_k z^{N-k}} \quad \text{where } q_0 = 1$$

If $q_0 \neq 1$, we can always divide by q_0 so that the leading denominator coefficient is 1. Hence, without loss of generality, $q_0 = 1$ is assumed.

If $X(z) = \dfrac{\sum_{\ell=r}^{M} p_\ell z^{-\ell}}{1 + \sum_{k=1}^{N} q_k z^{-k}}$, it can be written as

$$X(z) = z^{-r} \frac{P_1(z)}{Q(z)}$$ where there are no pole-zero cancellations.

The inverse z-transform of $\frac{P_1(z)}{Q(z)}$ and that of $\frac{P(z)}{Q(z)}$ differ only by a delay of 'r' samples.

Hence we will assume $p_0 \neq 0$ and $q_0 = 1$

First assume that all the roots are distinct

$$Q(z) = \prod_{k=1}^{N} (1 - q_k z^{-1})$$

$$X(z) = \frac{P(z)}{\prod_{k=1}^{N}(1-q_k z^{-1})} = \sum_{k=1}^{N} \frac{\boxed{A_k}}{1-q_k z^{-1}}$$

RESIDUE

(lookup the MATLAB command "residue")

$$A_k = \left. \frac{P(z)}{\prod_{\substack{l=1 \\ l \neq k}}^{N} (1-q_l z^{-1})} \right|_{z=q_k}$$

Example

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

$$= \frac{2}{1-z^{-1}} + \frac{-1}{1-\frac{1}{2}z^{-1}}$$

To get the inverse z-transform, we need RoC information.

Three choices:

(i) $|z| < 1/2$ (ii) $1/2 < |z| < 1$ (iii) $|z| > 1$

left-sided two-sided right-sided

(i) $-2u[-n-1] + (1/2)^n u[-n-1]$

(ii) $-2u[-n-1] - (1/2)^n u[n]$

(iii) $2u[n] - (1/2)^n u[n]$

The Final answer depends on which particular RoC is specified.

If $M \geq N$, we must first divide to get quotient and remainder.

Thus, $X(z)$ is transformed to the form

$$X(z) = \sum_{r=0}^{M-N} c_r z^{-r} + \underbrace{\sum_{k=1}^{N} \frac{A_k}{1-q_k z^{-1}}}_{\text{num. degree } M' < N}$$

Example

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = 2 + \frac{-1+5z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$= 2 + \frac{-9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

Based on RoC, three different time-domain sequences are possible

(i) $|z|>1$: $2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$

(ii) $\frac{1}{2} < |z| < 1$: $2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[-n-1]$

(iii) $|z| < 1/2$: $2\delta[n] + 9\left(\frac{1}{2}\right)^n u[-n-1] - 8u[-n-1]$

Repeated Roots:

$$X(z) = \frac{G \prod_{i=1}^{M} (1-a_i z^{-1})}{\prod_{q=1}^{Q} (1-b_q z^{-1}) \prod_{\ell=1}^{R} (1-\gamma_\ell z^{-1})^{\sigma_\ell}}$$

where $M < N = Q + \sum_{\ell=1}^{R} \sigma_\ell$

$$X(z) = \sum_{q=1}^{Q} \frac{A_q}{1 - b_q z^{-1}} + \sum_{\ell=1}^{R} \sum_{k=1}^{\sigma_\ell} \frac{C_{\ell,k}}{(1 - \nu_\ell z^{-1})^k}$$

ν_ℓ = root

σ_ℓ = multiplicity

$$A_q = X(z)(1 - b_q z^{-1}) \Big|_{z = b_q}$$

$$C_{\ell,k} = \frac{1}{(-\nu_\ell)^{\sigma_\ell - k}(\sigma_\ell - k)!} \frac{d^{\sigma_\ell - k}}{d\xi^{\sigma_\ell - k}} \left[X(\xi^{-1})(1 - \sigma_\ell \xi)^{\sigma_\ell} \right] \Bigg|_{\xi = \frac{1}{\nu_\ell}} \qquad k = 1, 2, \ldots, \sigma_\ell$$

Example

$$X(z) = \frac{12 - 22z^{-1} + 16z^{-2}}{(1 - 2z^{-1})^3} \qquad R = 1$$

$$\sigma_\ell = 3$$

$$C_{1,3} = \frac{1}{(-2)^0 0!} \frac{d^0}{d\xi^0} \left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^3} (1-2\xi)^3 \right] \Bigg|_{\xi = \frac{1}{2}} = 5$$

$$C_{1,2} = \frac{1}{(-2)^1 1!} \frac{d}{d\xi} \left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^3} (1-2\xi)^3 \right] \Bigg|_{\xi = \frac{1}{2}} = 3$$

$$C_{1,1} = \frac{1}{(-2)^2 2!} \frac{d^2}{d\xi^2} \left[\frac{12 - 22\xi + 16\xi^2}{(1 - 2\xi)^2} (1-2\xi)^3 \right] \Bigg|_{\xi = \frac{1}{2}} = 4$$

$$X(z) = \frac{4}{1 - 2z^{-1}} + \frac{3}{(1 - 2z^{-1})^2} + \frac{5}{(1 + 2z^{-1})^3}$$

To proceed further we need ROC information.

You will need results similar to the following:

$$\frac{(n+1)(n+2)\dots(n+M-1)\,a^{n}u[n]}{(M-1)!} \longleftrightarrow \frac{1}{(1-az^{-1})^{M}} \qquad |z|>|a| \qquad M \geq 2$$

Contour Integral Method

$$X(z) = \frac{1}{2\pi j}\oint_{C} X(z)\,z^{n-1}\,dz$$

$$= \sum_i \left[\text{residues of } X(z)\,z^{n-1} \text{ evaluated at the poles encircled by } C\right]$$

For multiple poles, say an m^{th} order pole at $z = z_0$, $X(z)z^{n-1}$ can be written as $\frac{\Gamma(z)}{(z-z_0)^m}$. The residue at z_0 is

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Gamma(z) \bigg|_{z=z_0}$$

To verify that the inversion integral does indeed give back $x[n]$, we proceed as follows:

$$\frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_C \left[\sum_{k=-\infty}^{\infty} x[n] z^{-k} \right] z^{n-1} dz$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi j} \oint_C z^{n-k-1} dz$$

Recall $\frac{1}{2\pi j} \oint_C z^n dz = \begin{cases} 1 & n=-1 \\ 0 & \text{otherwise} \end{cases}$

Hence,

$$\sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi j} \oint_C z^{n-k-1} dz = x[n]$$

Alternately,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Let $z = re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

$X(re^{j\omega})$ is a 2π-periodic function in ω and hence

$x[n]r^{-n}$ can be thought of as the Fourier Series coefficients!

Thus,

$$x[n]r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) e^{j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) r^{n} e^{j\omega n} d\omega$$

Let $\quad z = re^{j\omega}$

$$dz = j\underbrace{re^{j\omega}}_{z}d\omega$$

$$d\omega = \frac{dz}{jz}$$

We can thus convert the real-integral into a contour integral by invoking the principle of analytic continuation. Hence,

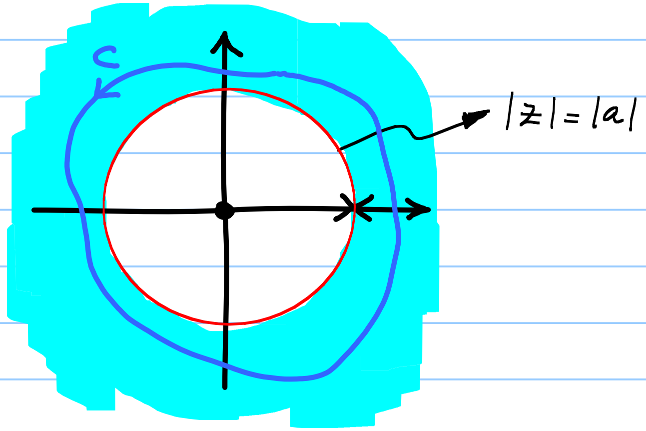
$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

as before.

EE5330 Aug. 28, 2013

Example

$$X(z) = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$



$$= \frac{z}{z-a}$$

$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z}{z-a} z^{n-1} dz$$

$$= \frac{1}{2\pi j} \oint \frac{z^n}{z-a} dz$$

For $n \geq 0$, the contour encloses one pole at $z = a$

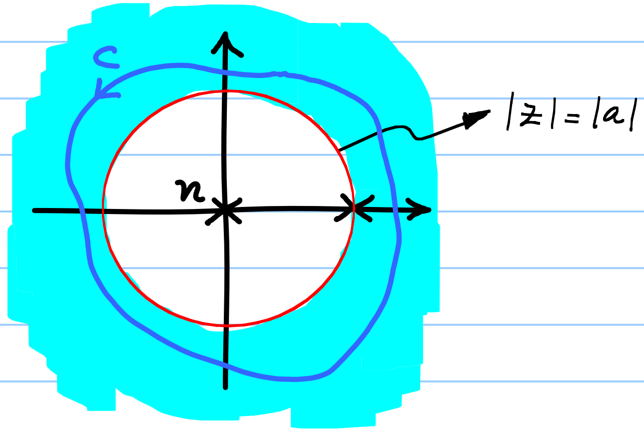
Residue at $z = a$: $(z-a) \dfrac{z^n}{z-a} \Big|_{z=a} = a^n$

For $n < 0$, $\dfrac{z^n}{z-a}$ can be written as $\dfrac{1}{z^n(z-a)}$ where $n > 0$

Hence, one can now easily see that C encloses not only the pole at $z = a$ but also an n^{th} order pole at $z = 0$.

Thus, residues have to be evaluated at $z = 0$ and $z = a$.

Residue at $z=a$: $(z-a)\dfrac{z^n}{z-a}\Big|_{z=a} = a^n$ $n<0$



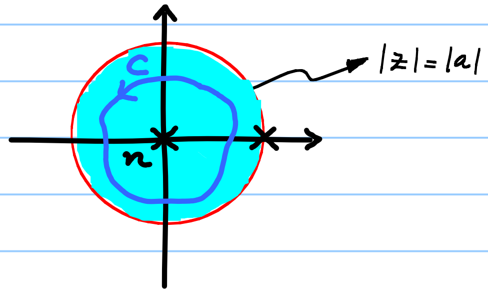
Residue at $z=0$: $\dfrac{1}{(n-1)!}\,\dfrac{d^{n-1}}{dz^{n-1}}\,\dfrac{1}{z-a}\Big|_{z=0}$

$= -a^n$

Hence, for $n<0$, sum of residues is zero.

Thus, $x[n] = a^n u[n]$

Repeat for $X(z) = \dfrac{1}{1-az^{-1}}$ with RoC $|z| < |a|$



Power Series Method

Examples:

(i) $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$

$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$

$\longleftrightarrow \{1, -\frac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{-1}, \frac{1}{2}\}$

(ii) $X(z) = \dfrac{1}{1-az^{-1}} \qquad |z| > |a|$

$= 1 + az^{-1} + a^2z^{-2} + \cdots$

$$\longleftrightarrow \{ \underset{\substack{\uparrow \\ n=0}}{1}, a, a^2, \dots \}$$

$$\begin{array}{r|l}
 & 1+az^{-1}+a^2z^{-2}+\cdots \\
\hline
1-az^{-1} & 1 \\
 & \underline{1-az^{-1}} \\
 & az^{-1} \\
 & \underline{az^{-1}-a^2z^{-2}} \\
 & a^2z^{-2} \\
 & \underline{a^2z^{-2}-a^3z^{-3}} \\
 & a^3z^3 \\
 & \vdots
\end{array}$$

(iii) $X(z) = \dfrac{1}{1-az^{-1}}$ $\quad |z|<|a|$

$$= \frac{-a^{-1}z}{1-a^{-1}z}$$

$$= -a^{-1}z\cdot\left(1+a^{-1}z+a^{-2}z^2+\cdots\right)$$

$$= -a^{-1}z-a^{-2}z^2-a^{-3}z^3-\cdots$$

$$\longleftrightarrow \{ \dots, -a^{-3}, -a^{-2}, a^{-1}, \underset{\uparrow}{0}, 0, 0, \dots \}$$

$$\begin{array}{r|l}
 & -a^{-1}z-a^{-2}z^2-a^{-3}z^3-\cdots \\
\hline
-az^{-1}+1 & 1 \\
 & \underline{1-a^{-1}z} \\
 & a^{-1}z \\
 & \underline{a^{-1}z-a^{-2}z^2} \\
 & a^{-2}z^2 \\
 & \underline{a^{-2}z^2-a^{-3}z^3} \\
 & a^{-3}z^3 \\
 & \vdots
\end{array}$$

(iv) $X(z) = \dfrac{1-a^2}{(1+a^2) - a(z+z^{-1})}$

If the RoC is $|a| < |z| < \dfrac{1}{|a|}$, then the corresponding $x[n]$ is $a^{|n|}$, i.e., it is two-sided.

If we carry out long-division directly, we will get a series expansion either in powers of z (anticausal sequence, corresponding to $|z| < |a|$) or in powers of z^{-1} (causal sequence, coresponding to $|z| > \dfrac{1}{|a|}$). We will not get the two-sided sequence.

To get the two-sided answer, we must proceed as follows:

$$X(z) = \underset{\substack{\downarrow \\ \text{causal part} \\ |z|>|a|}}{\frac{1}{1-az^{-1}}} + \underset{\substack{\downarrow \\ \text{anticausal part} \\ |z|<\frac{1}{|a|}}}{\frac{az}{1-az}}$$

Hence

$$\begin{aligned} X(z) &= 1 + az^{-1} + a^2 z^{-2} + \cdots \quad \text{causal part} \\ &\quad + az + a^2 z^2 + \cdots \quad \text{anticausal part} \end{aligned}$$

$$\longleftrightarrow \{ \ldots, a^2, a, \underset{\uparrow}{1}, a, a^2, a^3, \ldots \}$$

(v) $X(z) = e^{z}$

$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \qquad |z| < \infty$ [can also be stated as $z \in \mathbb{C}$]

$\longleftrightarrow \left\{ \ldots, \frac{1}{4!}, \frac{1}{3!}, \frac{1}{2!}, \frac{1}{1!}, \underset{\underset{n=0}{\uparrow}}{1}, 0, 0, 0, \ldots \right\}$

(vi) $\ln(1 + a z^{-1}) \qquad |z| > |a|$

Obtain the answer using both series expansion and the differentiation property.

The DTFT inversion formula can be derived from the z-transform inversion integral by substituting $z = e^{j\omega}$. The contour integral now becomes an integral over the real-valued variable 'ω'

$$z = e^{j\omega} \Rightarrow d\omega = \frac{dz}{jz}$$

Hence,

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$ IDTFT

Recall, the DTFT definition:

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}} \quad \text{DTFT}$$

Since $X(e^{j\omega})$ is a 2π-periodic function, the DTFT can be thought of as the Fourier Series expansion of $X(e^{j\omega})$ with $x[n]$ as the Fourier series coefficients. Hence the DTFT is nothing but Fourier series in disguise.

Examples

(i) $X(e^{j\omega}) = 2\pi\,\delta(\omega) \qquad -\pi \leq \omega \leq \pi$

$$= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$ valid for all ω

$$= 2\pi\,\tilde{\delta}(\omega)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\,\delta(\omega)\, e^{j\omega n}\, d\omega$$

$$= 1$$

$$\therefore \quad 1 \xleftrightarrow{DTFT} 2\pi\,\tilde{\delta}(\omega)$$

(ii) $e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\omega-\omega_0) \qquad -\pi \leq \omega < \pi$

which also follows from the modulation property

(iii) $\cos \omega_0 n \xleftrightarrow{DTFT} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \qquad -\pi \leq \omega < \pi$

(iv) $\sin \omega_0 n \xleftrightarrow{DTFT} \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \qquad -\pi \leq \omega < \pi$

(v) $x[n] = 1 \quad -N \leq n \leq N \xleftrightarrow{DTFT} \frac{\sin(2N+1)\omega/2}{\sin(\omega/2)}$

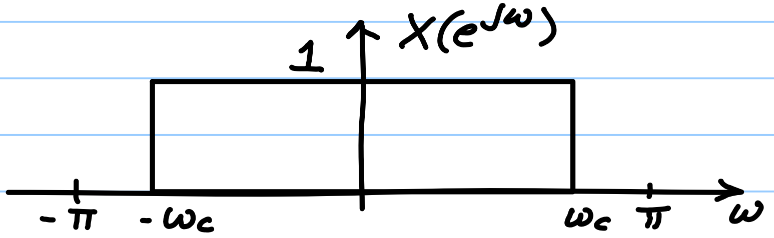
(vi) $X(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$

$$x[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 . \, d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$

Hence,

$$\frac{\sin \omega_c n}{\pi n} \xleftrightarrow{DTFT}$$

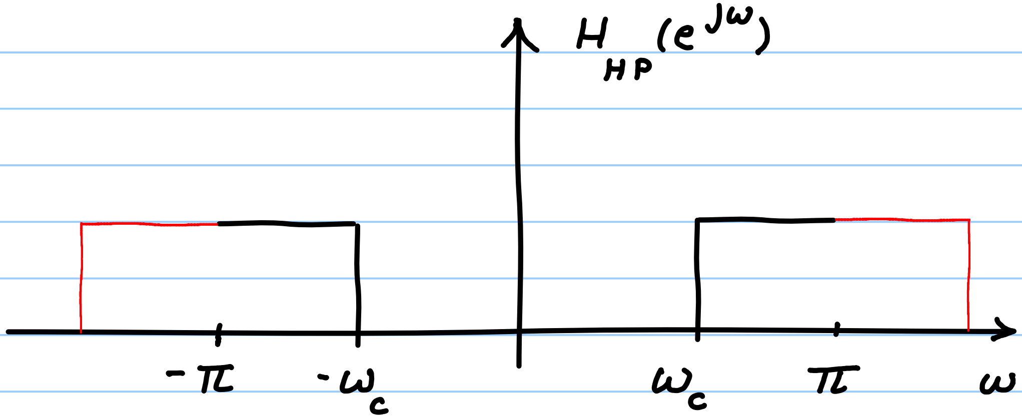


Ideal Lowpass Filter

Ideal HPF

$$H_{HP}(e^{j\omega}) = 1 - H_{LP}(e^{j\omega})$$

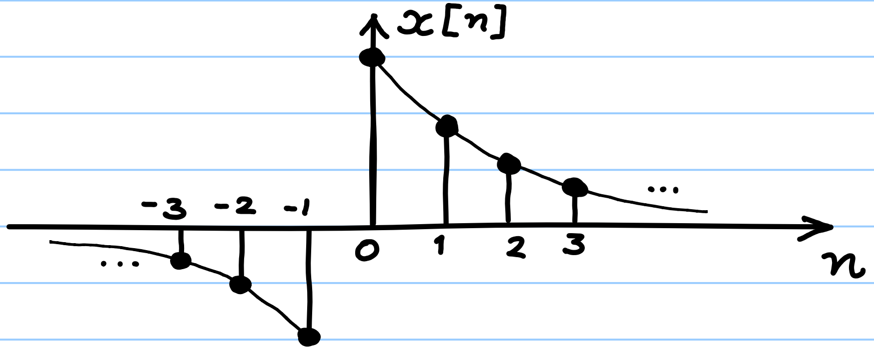
$$\longleftrightarrow \delta[n] - \frac{\sin \omega_c n}{\pi n}$$



Exercise

$$a^{|n|} \longleftrightarrow ?$$

Example



$$x[n] = a^{n} u[n] - a^{-n} u[-n-1]$$

$$\longleftrightarrow \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - a^{-1}e^{-j\omega}} \qquad |a| < 1$$

$$\underset{a \to 1}{Lt}\ x[n] = sgn[n] = \begin{cases} 1 & n \geqslant 0 \\ -1 & n < 0 \end{cases}$$

$$\underset{a \to 1}{Lt}\ X(e^{j\omega}) = \frac{2}{1 - e^{-j\omega}}$$

Thus, $sgn[n] \xleftrightarrow{DTFT} \dfrac{2}{1 - e^{-j\omega}}$

sgn[n] and u[n] are related as follows: $u[n] = \frac{1}{2} + \frac{1}{2} sgn[n]$

Hence

$$u[n] \xleftrightarrow{DTFT} \pi \tilde{\delta}(\omega) + \frac{1}{1 - e^{-j\omega}}$$

[Compare this with $u(t) \xleftrightarrow{CTFT} \pi \delta(\Omega) + 1/j\Omega$]

Some properties of the DTFT :

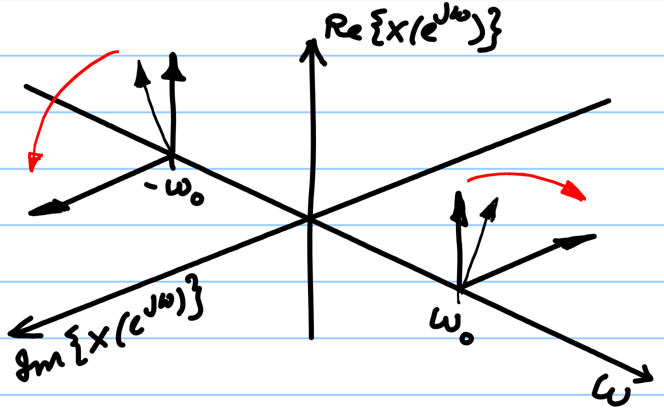
$$\sum_{n=-\infty}^{\infty} x[n]\, y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega$$

$$x[n]\, y[n] \longleftrightarrow \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)\, Y(\omega-\theta)\, d\theta}_{\text{circular convolution in the frequency domain}}$$

Note : $X(\omega)$ is used insted of $X(e^{j\omega})$

Derive the above two properties from the corresponding z-transform counterparts by substituting $z = e^{j\omega}$

$$x[n-n_0] \longleftrightarrow \underbrace{e^{-j\omega n_0}}_{\text{rotates } X(e^{j\omega}) \text{ by an angle } \omega n_0} X(e^{j\omega})$$



$$\cos \omega_0 n \longleftrightarrow \pi\left[\tilde{\delta}(\omega-\omega_0) + \tilde{\delta}(\omega+\omega_0)\right]$$

$$\sin \omega_0 n \longleftrightarrow \frac{\pi}{j}\left[\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)\right]$$

DTFT Symmetry Properties

$$x[n] = x_R[n] + j\,x_I[n]$$

$$X(\omega) = X_R(\omega) + j\,X_I(\omega)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R[n] \cos \omega n + x_I[n] \sin \omega n \right]$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} \left[x_I[n] \cos \omega n - x_R[n] \sin \omega n \right]$$

If $x[n] \in \mathbb{R}$, $X_R(-\omega) = X_R(\omega)$

$$X_I(-\omega) = -X_I(\omega)$$

$$|X(\omega)|^2 = X_R^2(\omega) + X_I^2(\omega)$$

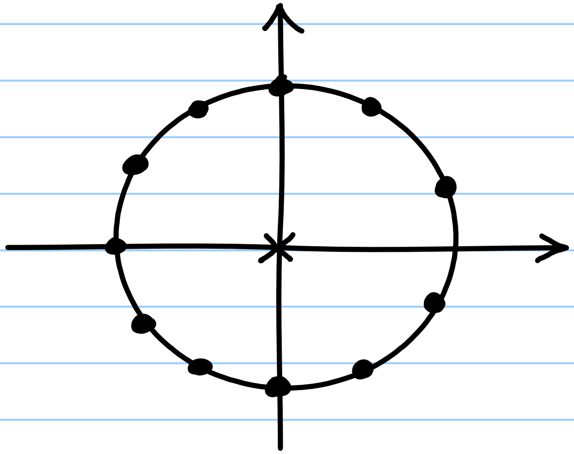
If $x[n]$ is real-valued, the magnitude of the DTFT is an even function of ω. The phase of the DTFT is an odd function of ω.

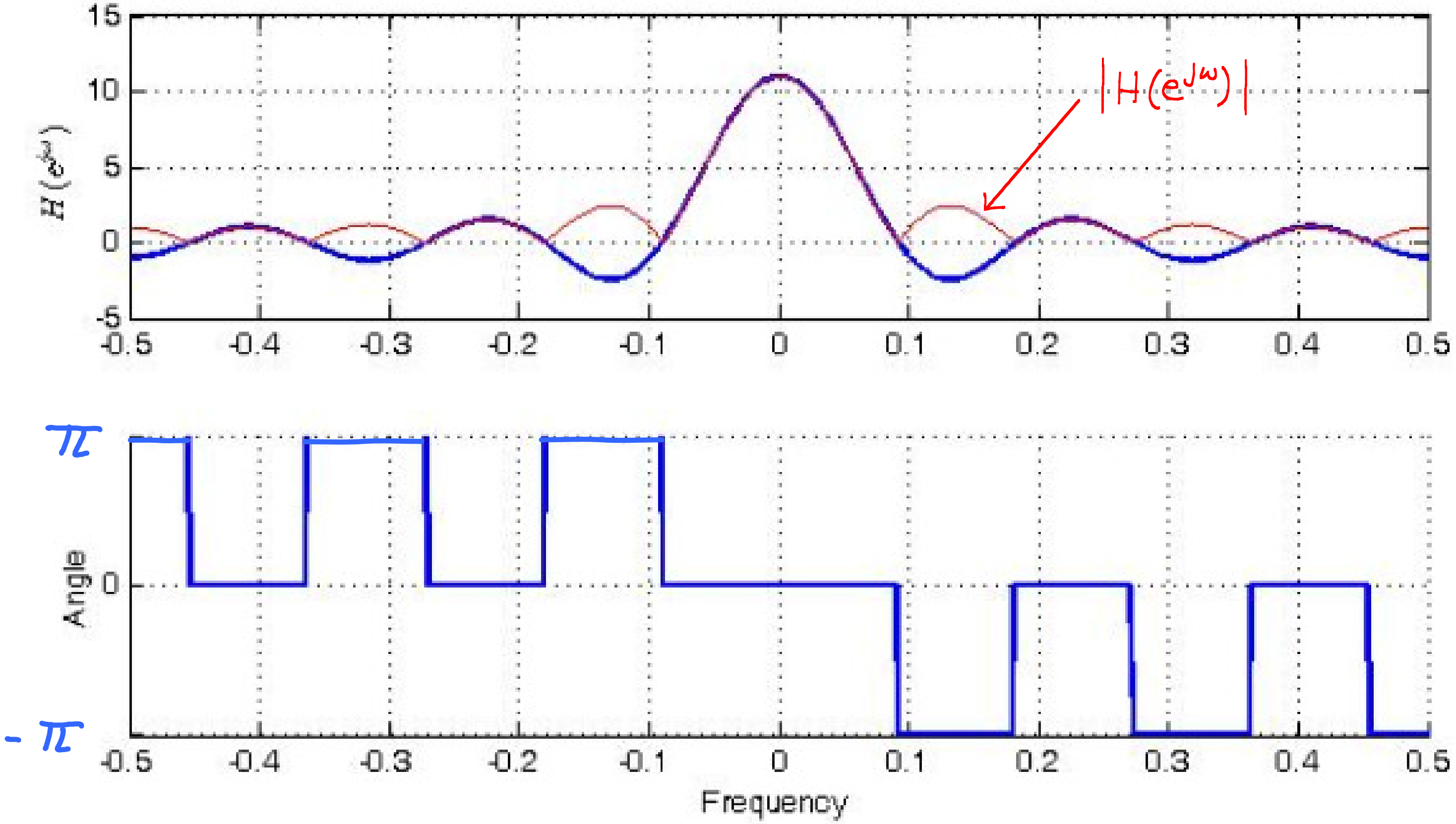
Recall that if $x[n] = x^*[n]$, then $X(\omega) = X^*(-\omega)$

Example

$$h[n] = 1 \qquad -M \le n \le M$$

$$H(e^{j\omega}) = \frac{\sin(2M+1)\omega/2}{\sin \omega/2}$$





15
10
5
0
-5
H (e^{jω})
|H(e^{jω})|
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
π
Angle
0
-π
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
Frequency

Exercise If $x[n] = jx_I[n]$, i.e., purely imaginary, then what symmetry, if any, does the DTFT possess ?

Exercise

$$A \cos(\omega_o n + \phi) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

Let $h[n]$ be real-valued. Then, $H(e^{j\omega}) = H^*(e^{-j\omega})$

$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$ where $\theta(\omega) = \measuredangle H(e^{j\omega})$

Show that $y[n] = A\,|H(e^{j\omega_o})|\, \cos(\omega_o n + \phi + \theta(\omega_o))$

Is the above $x[n]$ an eigensignal ?

Stability

An LTI system is BIBO stable if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h[n] \cdot e^{j\omega n}| < \infty$$

i.e., unit circle is part of RoC.

Causality

For a causal system with transfer function $H(z)$, the RoC is of the form $|z| > r_{max}$, where r_{max} is the radius of the furthermost pole (we have assumed $H(z)$ is rational).

When is a causal system stable?

Consider the furthermost pole. In the partial fraction expansion, it will give rise to (assuming simple pole)

$$\frac{A_k}{1 - p_k z^{-1}} \longleftrightarrow A_k (p_k)^n u[n]$$

$$\Rightarrow \sum_{n=0}^{\infty} |h[n]| < \infty \quad \boxed{\text{iff}} \quad |p_k| < 1$$

$\Rightarrow$ **all poles must lie inside the unit circle**

Since $r_{max} < 1$, the unit circle is now part of the RoC, which condition must be satisfied for BIBO stability.

If p_k is not a simple pole,

$$\frac{(n+1)(n+2)\cdots(n+M-1)}{(M-1)!}\, a^n u[n] \longleftrightarrow \frac{1}{(1-p_k z^{-1})^M} \qquad |z| > |p_k|$$

$$\sum_{n=0}^{\infty} n^{\ell} |p_k|^n < \infty \quad \boxed{\text{iff}} \quad |p_k| < 1 \quad \text{for ANY } \ell$$

Hence all poles must lie strictly inside the unit circle for a causal system.

For an anticausal system, the RoC is of the form $|z| < r_{min}$, where r_{min} is the radius of the innermost pole. In this case, for stability, all poles must lie strictly outside the unit circle. Once again the unit circle is part of the RoC, which is essential for BIBO stability.

EE 5330 Sep. 3, 2013

Paley-Wiener Theorem

Let $h[n] = 0$ for $n < 0$ and let $h[n] \in l_2$. Let $h[n]$ possess DTFT $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$. Then

$$\int_{-\pi}^{\pi} |\ln |H(e^{j\omega})|| \, d\omega < \infty$$

Conversely, if $|H(e^{j\omega})| \in \mathcal{L}_2[-\pi, \pi)$ and $\int_{-\pi}^{\pi} |\ln |H(e^{j\omega})|| \, d\omega < \infty$, then there exists $\theta(\omega)$ s.t. the filter with transfer function $H(e^{j\omega}) = |H(e^{j\omega})| \cdot e^{j\theta(\omega)}$ has an impulse response that is causal.

Observations

(i) $H(e^{j\omega})$ cannot be zero over an interval.

(ii) $H(e^{j\omega})$ cannot be constant over an interval.

(iii) The transition from passband to stopband cannot be abrupt.

(iv) The real and imaginary parts of $H(e^{j\omega})$ cannot be independent.

To see how the real and imaginary parts of $H(e^{j\omega})$ are related, we proceed as follows.

Any $h[n]$ can be written as

$$h[n] = h_e[n] + h_o[n]$$

where

$$h_e[n] = \frac{h[n] + h[-n]}{2} \qquad h_o[n] = \frac{h[n] - h[-n]}{2}$$

If $h[n] = 0$ for $n < 0$, $h[n]$ and $h[-n]$ do not overlap except at $n = 0$. Hence, $h[n]$ can be recovered from $h_e[n]$ as follows:

$$h[n] = 2h_e[n]u[n] - h_e[n]\delta[n]$$

$$= 2h_e[n]u[n] - h[0]\delta[n] \qquad (\because h_e[0] = h[0])$$

OTOH, since $h_o[0] = 0$ always, we can recover $h[n]$ from $h_o[n]$ for $n > 0$ only. $h[0]$ information is needed for full recovery.

Recall $h[n] \longleftrightarrow H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$

$$h_e[n] = \frac{h[n] + h[-n]}{2} \longleftrightarrow \frac{H(e^{j\omega}) + H(e^{-j\omega})}{2}$$

If $h[n] \in \mathbb{R}$, then $H(e^{-j\omega}) = H^*(e^{j\omega})$. Hence,

$$h_e[n] \longleftrightarrow H_R(e^{j\omega})$$

Also recall

$$u[n] \longleftrightarrow \pi\,\tilde{\delta}(\omega) + \frac{1}{1 - e^{-j\omega}}$$

Hence,

$$2\,h_e[n]\,u[n] \longleftrightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\left[\pi\,\tilde{\delta}(\omega-\theta) + \frac{1}{1-e^{-j\overline{\omega-\theta}}}\right]d\theta$$

$$= H_R(e^{j\omega}) + \frac{1}{\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\,\frac{1}{1-e^{j\overline{\omega-\theta}}}\,d\theta$$

But

$$\frac{1}{1-e^{-j\omega}} = \frac{1-\cos\omega - j\sin\omega}{2-2\cos\omega} = \frac{1}{2} - \frac{j}{2}\cot\left(\frac{\omega}{2}\right)$$

Hence,

$$2h_e[n]\,u[n] \longleftrightarrow H_R(e^{j\omega}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\,d\theta - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta$$

$$= H_R(e^{j\omega}) + \underbrace{h_e[0]}_{h[0]} - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

Hence

$$2h_e[n]u[n] - h[0] = h[n] \longleftrightarrow H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega}) = H_R(e^{j\omega}) - \frac{j}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

Hence, equating the real and imaginary parts, we get,

$$\boxed{H_I(e^{j\omega}) = \frac{-1}{2\pi}\int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta}$$

Similarly, one can show

$$\boxed{H_R(e^{j\omega}) = h[0] + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_I(e^{j\theta})\cot\left(\frac{\omega-\theta}{2}\right)d\theta}$$

The above are called the Discrete Hilbert Transform relationships.

Thus, for a causal sequence, the real and imaginary parts of $H(e^{j\omega})$ are not independent. If the real and imaginary parts are related, does it imply that the magnitude and phase are also related ?

EE 5330 Sep. 4, 2013

The integrals in the DHT relationships are all Principal Value integrals, i.e.,

$$H_I(e^{j\omega}) = \frac{-1}{2\pi} \text{ P.V.} \int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

$$= \frac{-1}{2\pi} \lim_{\epsilon \to 0} \left[\int_{-\pi}^{\omega-\epsilon} (\,\cdot\,)\, d\theta + \int_{\omega+\epsilon}^{\pi} (\,\cdot\,)\, d\theta \right]$$

Stability

Let $H(z) = \frac{B(z)}{A(z)} = \frac{B(z)}{\prod_{k=1}^{N}(1-p_k z^{-1})}$

We know that, if the system is causal, then for stability we require

$|p_k| < 1 \quad \forall k.$

Tests have been devised to check if $|p_k| < 1 \quad \forall k$ without explicit roots computation.

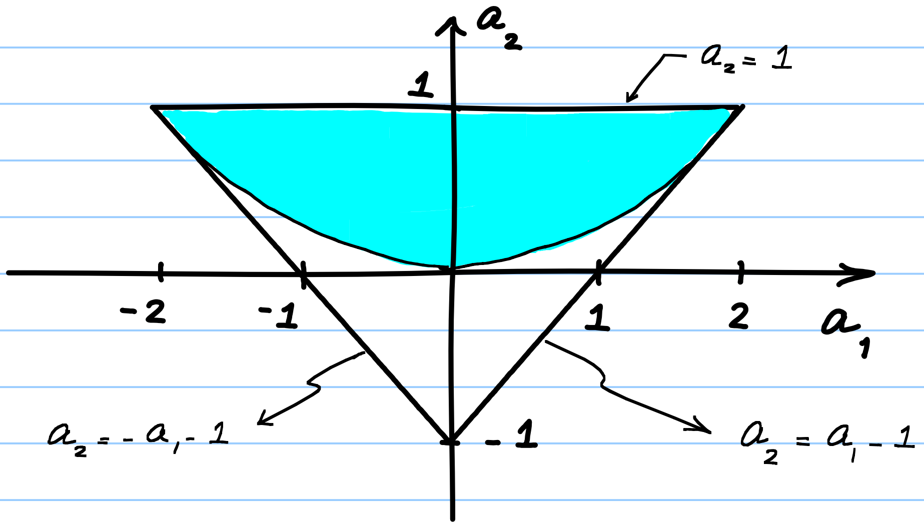
For 2^{nd} order systems, we will show that the conditions to be satisfied are :

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$$

1) $|a_2| < 1 \Rightarrow -1 < a_2 < 1$

2) $|a_1| < 1 + a_2 \Rightarrow a_1 < 1 + a_2$

$$-a_1 < 1 + a_2$$



These conditions are satisfied in the triangular region shown on the right, the so-called stability triangle.

In the shaded region, the roots occur in complex conjugate pairs.

We will consider the case of complex conjugates roots first.

$$A(z) = (1 - re^{j\omega_0} z^{-1})(1 - re^{-j\omega_0} z^{-1})$$

$$= 1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}$$

$$= 1 + a_1 z^{-1} + a_2 z^{-2}$$

Stability demands that

$$|a_2| = r^2 < 1$$

$$|a_1| = |2r \cos \omega_0| < 1 + r^2$$

For stability, roots must lie inside the unit circle (assuming causality)

Hence $r < 1 \Rightarrow |a_2| = r^2 < 1$, i.e., the first condition is satisfied.

$$0 < r < 1 \iff (1-r)^2 > 0 \Rightarrow 2r < 1+r^2$$

$$0 < r < 1 \iff (1+r)^2 > 0 \Rightarrow -2r < 1+r^2$$

Hence $|2r| < 1+r^2 \Rightarrow |2r\cos\omega_0| < 1+r^2$

Thus we have shown that the stability conditions are satisfied iff the complex conjugate roots are inside the unit circle.

Now consider $A(z) = (1-r_1z^{-1})(1-r_2z^{-1})$ where $-1 < r_i < 1$

$$-1 < r_i < 1 \implies 0 < 1 + r_i < 2$$

$$1 > -r_i > -1 \implies 0 < 1 - r_i < 2$$

Hence

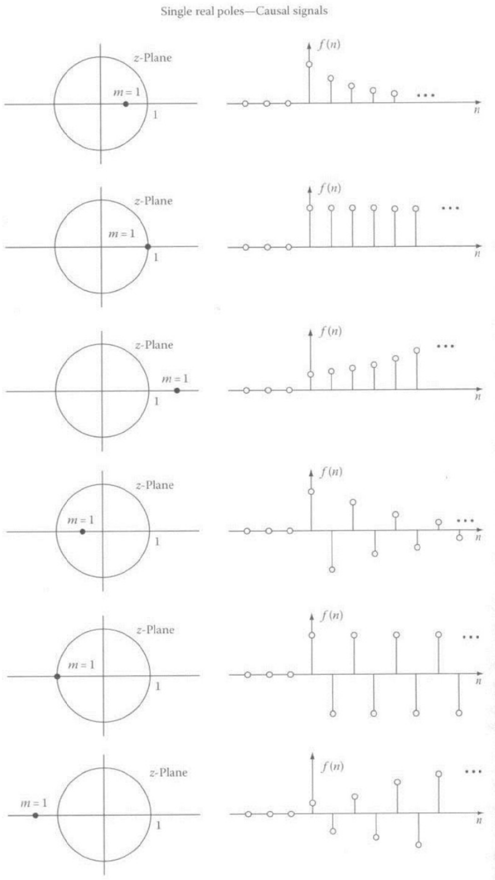
$$A(1) = 1 + a_1 + a_2 = (1 - r_1)(1 - r_2) > 0$$

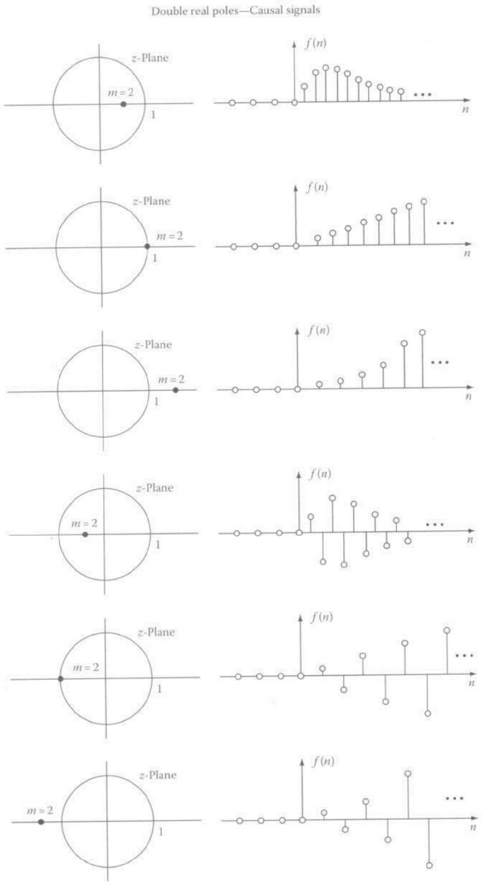
$$A(-1) = 1 - a_1 + a_2 = (1 + r_1)(1 + r_2) > 0$$

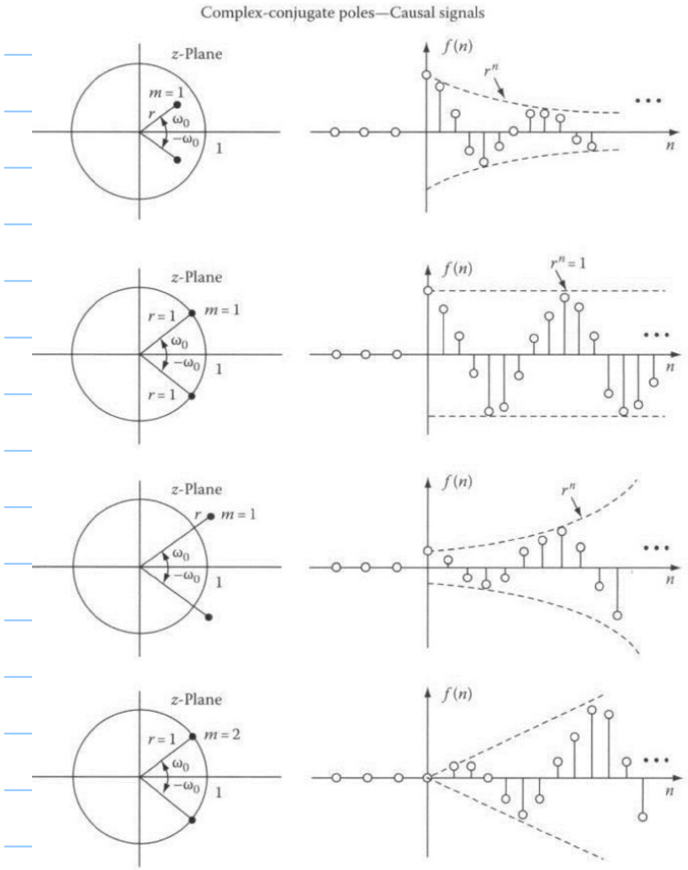
Thus,
$$\left.\begin{array}{l} -a_1 < 1 + a_2 \\ a_1 < 1 + a_2 \end{array}\right\} \implies |a_1| < 1 + a_2$$

Hence, once again, the conditions are satisfied iff the real-valued roots are inside the unit circle.

Some Typical Impulse Responses


Single real poles—Causal signals


Double real poles—Causal signals


Complex-conjugate poles—Causal signals

From "Transforms and Applications Handbook", Alexander Poularikas (Ed.), 3rd edition, CRC Press, 2010

EE5330 Sep. 5, 2013

What modes are present in the output when an input is applied?

Let $X(z) = \frac{P(z)}{Q(z)}$. Assume, for illustrative purposes, only simple poles are present in $X(z)$. Then,

$$X(z) = \frac{P(z)}{Q(z)} = \sum_{k=1}^{Q} \frac{A_k}{1-\xi_k z^{-1}} \longleftrightarrow \sum_{k=1}^{Q} A_k (\xi_k)^n u[n]$$

(assuming causality)

$(\xi_k)^n u[n]$ are called as the input modes.

Similarly, let $H(z) = \dfrac{B(z)}{A(z)} = \displaystyle\sum_{k=1}^{N} \frac{B_k}{1-p_k z^{-1}} \longleftrightarrow \sum_{k=1}^{N} B_k (p_k)^n u[n]$

(again assuming causality)

$(p_k)^n u[n]$ are called as the natural modes (also called system modes)

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = \frac{P(z)}{Q(z)} \frac{B(z)}{A(z)}$$

$$= \sum_{\ell=1}^{Q} \frac{C_\ell}{1-\xi_\ell z^{-1}} + \sum_{k=1}^{N} \frac{D_k}{1-p_k z^{-1}}$$

Hence, assuming causality,

$$y[n] = \underbrace{\sum_{\ell=1}^{Q} C_\ell (\xi_\ell)^n u[n]}_{\text{input modes}} + \underbrace{\sum_{k=1}^{N} D_k (p_k)^n u[n]}_{\text{natural modes}}$$

The output consists of input modes and natural modes.

Input modes are the particular solution

Natural modes are the homogeneous solution

Similar arguments apply for CT systems governed by LCCDE

$$Y(s) = \frac{P(s)}{Q(s)} \cdot \frac{B(s)}{A(s)} \leftrightarrow y(t) = \text{input modes} + \text{natural modes}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = a^n u[n]$$

$$x[n] = b^n u[n] \qquad b \neq a$$

$$Y(z) = X(z) H(z)$$

$$= \frac{1}{1-az^{-1}} \quad \frac{1}{1-bz^{-1}}$$

$$= \frac{1}{a-b}\left[\frac{a}{1-az^{-1}} - \frac{b}{1-bz^{-1}}\right]$$

$$\longleftrightarrow \frac{a}{a-b}\, a^{n} u[n] - \frac{b}{a-b}\, b^{n} u[n]$$

natural mode ⟶ ↑ ↑ ⟵ input mode

If $x[n] = a^{n} u[n]$, $y[n] = (n+1) a^{n} u[n]$ ← RESONANCE !

Typical 2nd Order Section

If $H(z) = \dfrac{B(z)}{A(z)} = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$

where $a_k, b_k \in \mathbb{R}$. The following is a typical pair, assuming simple poles:

$$\frac{A_k}{1 - p_k z^{-1}} + \frac{A_k^*}{1 - p_k^* z^{-1}}$$

$$= \frac{(A_k + A_k^*) - z^{-1}(A_k^* p_k + A_k p_k^*)}{1 - (p_k + p_k^*) z^{-1} + |p_k|^2 z^{-2}}$$

$$= \frac{p_0 + p_1 z^{-1}}{1 + q_1 z^{-1} + q_2 z^{-2}} \qquad p_k, q_k \in \mathbb{R}$$

The above is a typical second order section that shows up in practice in the parallel form implementation of digital filters.

Another popular form:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 \prod_{\ell=1}^{M} (1 - z_\ell z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

Typical 2^{nd} order section: $\dfrac{c_0 + c_1 z^{-1} + c_2 z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}} \qquad c_k, d_k \in \mathbb{R}$

where two complex-conjugate roots have been combined – Cascade form section.

One-sided z-Transform:

The two-sided z-transform cannot be used for solving LCCDE with initial conditions. The one-sided z-transform is naturally equipped to do so.

$$X_+(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \qquad \text{RoC}: |z| > r_{max}$$

The time-shift property behaves differently when compared with its two-sided counterpart.

Let $k \geq 0$ and let $y[n] = x[n-k]$. It is easy to see that $Y_+(z) = z^{-k} X_+(z)$. This is identical to the result of the two-sided counterpart.

OTOH, consider $y[n] = x[n+k]$ where $k > 0$. Then,

$$x : \{ \ldots 0, 0, 0, \underset{\uparrow}{x[0]}, x[1], x[2], x[3], \ldots \}$$

$$y : \{ \ldots 0, 0, 0, x[0], x[1], \ldots, x[k-1], \underset{\uparrow}{x[k]}, x[k+1], x[k+2], \ldots \}$$

$$X_+(z) = x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$Y_+(z) = x[k] + x[k+1]z^{-1} + x[k+2]z^{-2} + \cdots$$

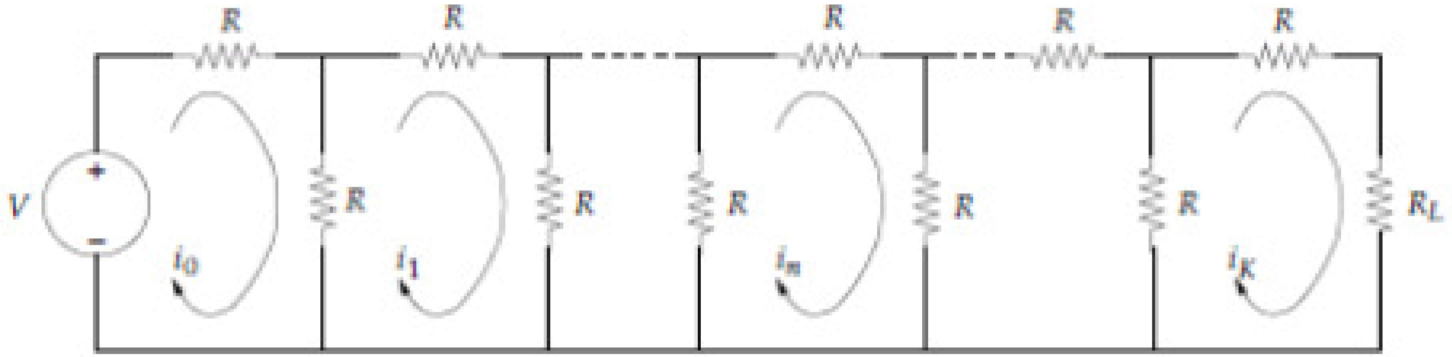
$$X_+(z) - \sum_{n=0}^{k-1} x[n]z^{-n} = x[k]z^{-k} + x[k+1]z^{-(k+1)} + \cdots$$

Thus,

$$Y_+(z) = z^k\left[X_+(z) - \sum_{n=0}^{k-1} x[n]z^{-n}\right]$$

Note that if all the initial conditions are zero, the above reduces to the earlier result.

Example The one-sided transform can be used for solving the currents in the circuit shown below:



The difference equation that relates the loop currents i_n, i_{n+1}, i_{n+2} can easily be verified to be the following:

$$i_n - 3i_{n+1} + i_{n+2} = 0$$

Transforming the above, we get,

$$I(z) - 3z\left[I(z) - i_0\right] + z^2\left[I(z) - i_0 - i_1 z^{-1}\right] = 0$$

$$\Rightarrow \quad I(z) = \frac{z(i_0 z - 3i_0 + i_1)}{z^2 - 3z + 1}$$

We can eliminate i_1 from the equation related to the first loop:

$$V = 2Ri_0 - i_1 R \Rightarrow i_1 = 2i_0 - \frac{V}{R}$$

$$i_n = i_0\left[\cosh \omega_0 n + \frac{\frac{1}{2} - (V/Ri_0)}{\sqrt{5}/2} \sinh \omega_0 n\right] \quad \text{where} \quad \cosh \omega_0 = \frac{3}{2} \quad \sinh \omega_0 = \frac{\sqrt{5}}{2}$$

Note on the convergence condition of the DTFT

Recall the following definition:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

If $|H(e^{j\omega})| < \infty$, then $\left| \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |h[n]|$

Thus, the DTFT exists if the sequence is absolutely summable.

This condition is sufficient but not necessary. Sequences such as $u[n]$ are not absolutely summable but yet possess DTFT.

If the sequence is absolutely summable, the DTFT will be a continuous function of ω. [Why?]

EE 5330 Sep. 12, 2013

Since the DTFT is nothing but the Fourier series expansion of the 2π-periodic frequency domain function, the following mean-square convergence theorem for Fourier Series is applicable.

The series $\sum_{n=-N}^{N} x[n]e^{-j\omega n}$ converges to $X(e^{j\omega})$ in the mean-square sense if $X(e^{j\omega})$ is square integrable over $[-\pi,\pi)$, i.e., $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega < \infty$.

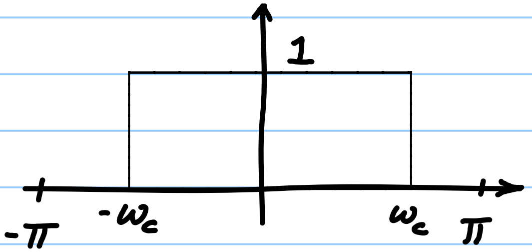
Let $X_N(e^{j\omega}) = \sum_{n=-N}^{N} x[n]e^{-j\omega n}$. MS convergence means

$$\int_{-\pi}^{\pi} |X_N(e^{j\omega}) - X(e^{j\omega})|^2 d\omega \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Note that if $X(e^{jw})$ is square-integrable, then $x[n] \in \ell_2$ [why?]

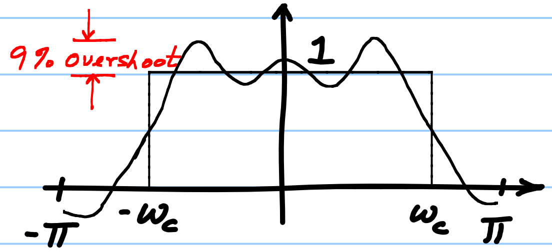
The lack of pointwise convergence but only mean-square convergence is illustrated through Gibbs phenomenon

$$\frac{\sin \omega_c n}{\pi n} \longleftrightarrow$$



$$\frac{\sin \omega_c n}{\pi n} \longleftrightarrow$$

$$-N \leq n \leq N$$



Relationship Between Laplace & Z-transforms

Let $x(t) \xleftrightarrow{\mathcal{L}} X(s)$

Define $$x_p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$$X_p(s) = \mathcal{L}\{x_p(t)\} = \int_{-\infty}^{\infty} x_p(t)e^{-st}dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,e^{-snT}$$

Recall $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

By letting $x(nT) \equiv x[n]$, we see that

$X_p(s)\Big|_{z=e^{sT}} = X(z)$

Note that, since $e^{sT} = e^{(s+j\frac{2\pi}{T}n)T}$, $X_p(s+j\frac{2\pi n}{T}) = X_p(s)$

The mapping e^{sT} maps (a) the left half of the s-plane to inside the unit circle, (b) the $j\Omega$ axis to the unit circle, and (c) the right half of the s-plane to outside the unit circle.

Horizontal lines in the s-plane get mapped to radial lines in the z-plane

Vertical lines in the s-plane get mapped to circles in the z-plane

$\Rightarrow$ vertical strips get mapped to annular regions.

The s-plane origin, i.e., $s=0$, gets mapped to $z=1$

Note that $s=\frac{1}{T}\ln z$. Since ln is a multivalued function, a single point $z_1=r_1e^{j\theta_1}$ gets mapped to an infinite number of points, i.e., $s=\frac{1}{T}\ln re^{j\theta_1}=\frac{1}{T}\ln re^{j(\theta_1+2n\pi)}=\frac{1}{T}\ln r+\frac{1}{T}j(\theta+2\pi n)$

EE5330 Sep. 16, 2013

Frequency Response of Systems with Rational Transfer Function:

Frequency selective filtering is very important in many practical applications. We can obtain the frequency response by

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

provided the unit circle is part of the RoC, i.e., $e^{j\omega} \in$ RoC

If $e^{j\omega} \in$ RoC, the system is also BIBO stable.

In practice, we will concern ourselves with causal and stable systems. In particular, we will restrict ourselves to the class of

LTI systems characterized by LCCDE.

Some important frequency responses are: LPF, HPF, BPF, BSF, differentiator, and Hilbert transformer.

If the system is to be causal, then ideal, brickwall filters cannot be realized, since they violate the Paley-Wiener theorem.

We will approximate the ideal responses using rational transfer functions, i.e., by systems that are realizable.

Consider $y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{\ell=0}^{M} b_\ell\, x[n-\ell]$

Taking z-transforms and simplifying,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1 + \sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

In product form,

$$H(z) = b_0 \frac{\prod_{\ell=1}^{M} (1 - z_\ell z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} = b_0\, z^{\overbrace{N-M}^{\uparrow\ \text{|N-M| order trivial pole or zero}}} \frac{\prod_{\ell=1}^{M} (z - z_\ell)}{\prod_{k=1}^{N} (z - p_k)}$$

Since the system is stable, $e^{j\omega} \in$ RoC. Hence,

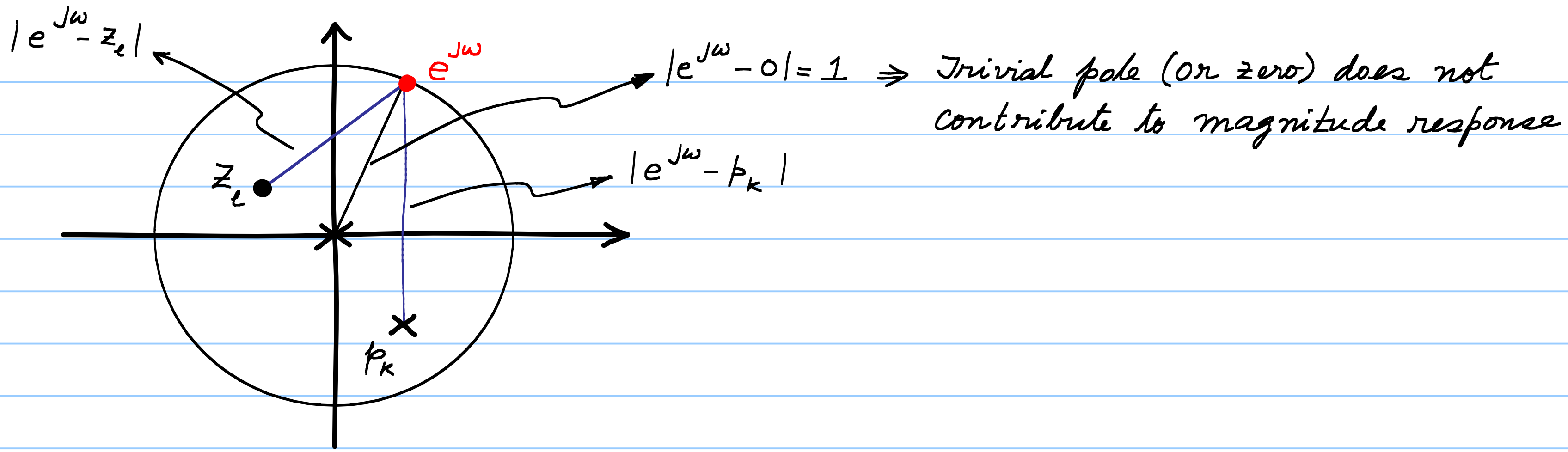
$$H(e^{j\omega}) = b_0 \, e^{j\omega(N-M)} \, \frac{\prod_{\ell=1}^{M} (e^{j\omega} - z_\ell)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)} = \underbrace{|H(e^{j\omega})|}_{\text{magnitude response}} \, e^{j \overbrace{\measuredangle H(e^{j\omega})}^{\text{phase response}}}$$

$$|H(e^{j\omega})| = |b_0| \, \cancelto{1}{\left|e^{j\omega(N-M)}\right|} \, \frac{\prod_{\ell=1}^{M} |e^{j\omega} - z_\ell|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

Because a pole or zero at $z=0$ does not contribute to the magnitude frequency response, they are called TRIVIAL pole/zero

Trivial poles and zeros contribute to the phase response.

Consider $|e^{j\omega} - z_\ell|$. Geometrically, this denotes the distance from $e^{j\omega}$ (point on the unit circle) to z_ℓ (zero at $z = z_\ell$). Thus, the numerator term is the product of all the distances from $e^{j\omega}$ to all the zeros. Similarly, the denominator is the product of all the distances from $e^{j\omega}$ to all the poles. Finally, $|H(e^{j\omega})|$ is the ratio of these two product of distances, multiplied by the gain term $|b_0|$. $|H(e^{j\omega})|$ changes as 'ω' changes.



Because $|H(e^{j\omega})|$ spans a large range, we plot the magnitude on a log scale. In particular, we plot $20 \log_{10} |H(e^{j\omega})|$ (or, equivalently, $10 \log_{10} |H(e^{j\omega})|^2$). The gain term $|b_0|$ merely shifts the curve up or down in the log scale.

The same geometric interpretation holds good in the s-plane also, when interpreting the magnitude of $H(s)\big|_{s=j\Omega}$. For rational $H(s)$,

$$|H(j\Omega)| = |b_0| \frac{\prod_{l=1}^{M} |j\Omega - z_l|}{\prod_{k=1}^{N} |j\Omega - p_k|}$$

$|H(j\Omega)|$ is the ratio of the product of all the distances from $j\Omega$ to all the zeros to product of all the distances from $j\Omega$ to all the poles, multiplied by $|b_0|$.

The above geometric interpretation reveals that there is no point in the s-plane that is at a constant distance as we move along the $j\Omega$ axis. Hence there is no concept of trivial pole in the s-plane (unlike in the z-plane, where the origin is at a constant distance as we move along the unit circle).

EE5330 Sep. 17, 2013

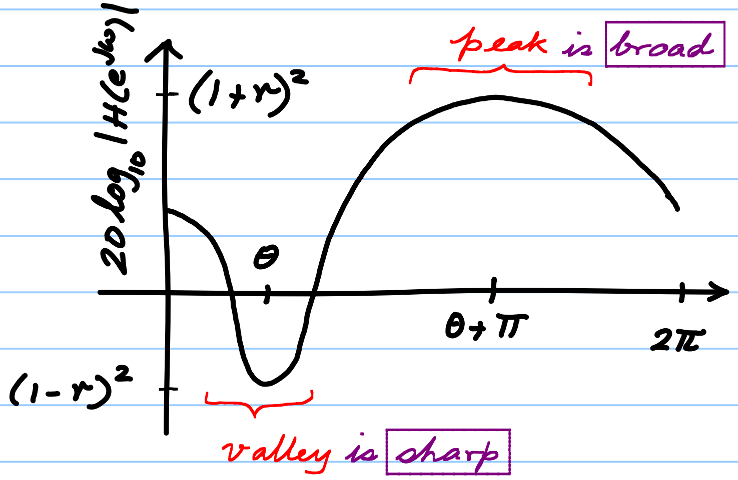
Response of a single complex zero:

Let $H(z) = 1 - re^{j\theta}z^{-1}$

$$|H(e^{j\omega})|^2 = |1 - re^{j\theta}e^{-j\omega}|^2$$

$$= 1 + r^2 - 2r\cos(\omega - \theta)$$

replacing 'w' by '-w' gives a different response as h[n] is complex-valued, except when $\omega = 0$ and $\omega = \pi$



Minimum occurs at $\omega = \theta$; $|H(e^{j\omega})|^2_{min} = (1-r)^2$

Maximum occurs at $\omega = \theta + \pi$; $|H(e^{j\omega})|^2_{max} = (1+r)^2$

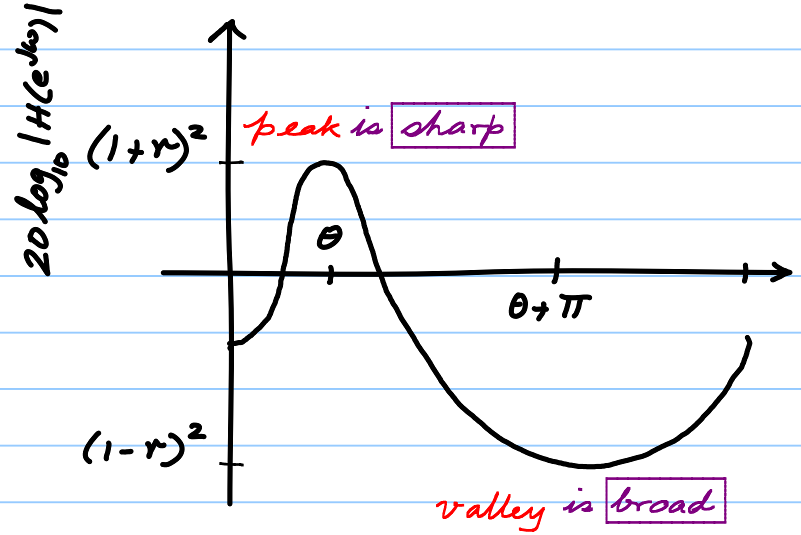
For $r = 0.9$, $|H(e^{j\omega})|^2_{min} = 0.01$

$|H(e^{j\omega})|^2_{max} = 3.61$

For a single complex pole,

$$H(z) = \frac{1}{1 - re^{j\theta}z^{-1}}$$

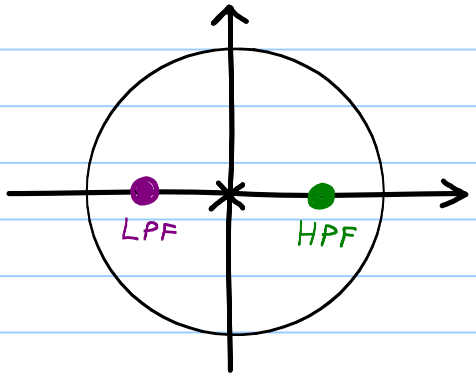
$\Rightarrow$ the log plot of $|H(e^{j\omega})|^2$ is the negative of the previous plot

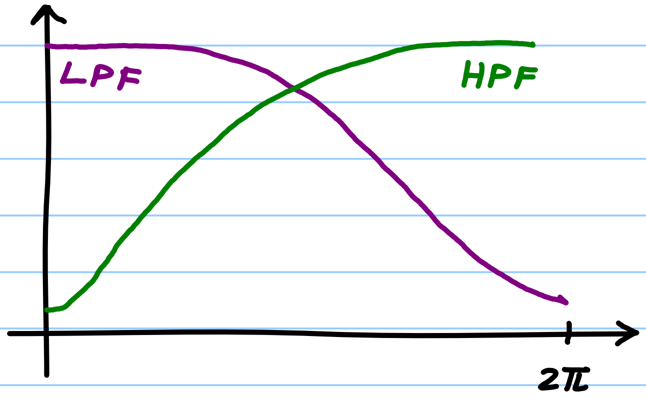


Pole near the unit circle boosts the frequency response

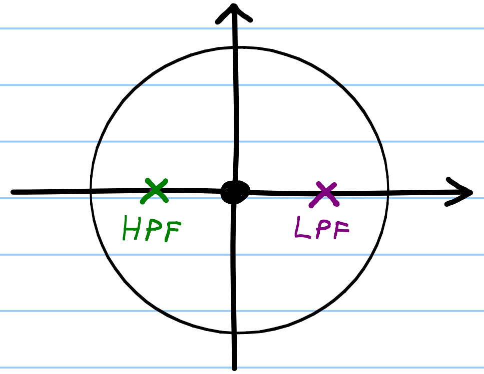
Zero near the unit circle attenuates the frequency response

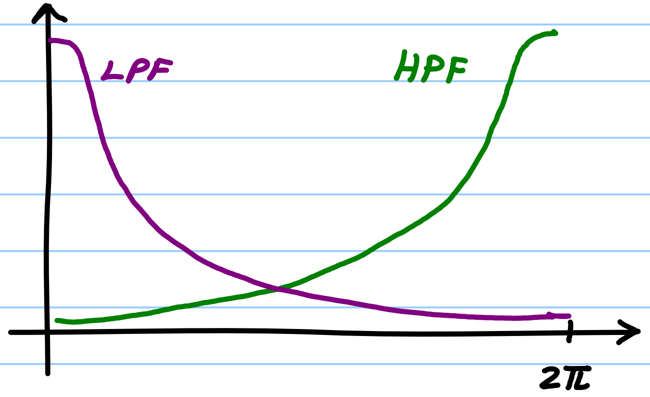
Lowpass and Highpass filters realized using single complex zero :





Lowpass and Highpass filters realized using single complex pole:





The main difference between an LPF realized using a pole versus another realized using a zero is the narrowness of the passband.

Zeros cause sharp valleys and broad peaks in the frequency response

Poles cause sharp peaks and broad valleys in the frequency response

Consider the following two LPFs :

(1) Realized Using a Pole :

$$H(z) = \frac{1}{1 - a z^{-1}} \qquad 0 < a < 1$$

$$|H(e^{j0})|^2 = \frac{1}{(1-a)^2}$$

$$= 100 \ (20 dB)$$

if $a = 0.9$

$$|H(e^{j\frac{\pi}{2}})|^2 = \frac{1}{1+a^2} = 0.55$$

(ii) Realized Using a zero :

$$H(z) = 1 + a z^{-1} \qquad 0 < a < 1$$

$$|H(e^{j0})|^2 = (1+a)^2$$

$$= 3.61 \ (5.58 dB)$$

if $a = 0.9$

$$|H(e^{j\frac{\pi}{2}})|^2 = 1 + a^2$$

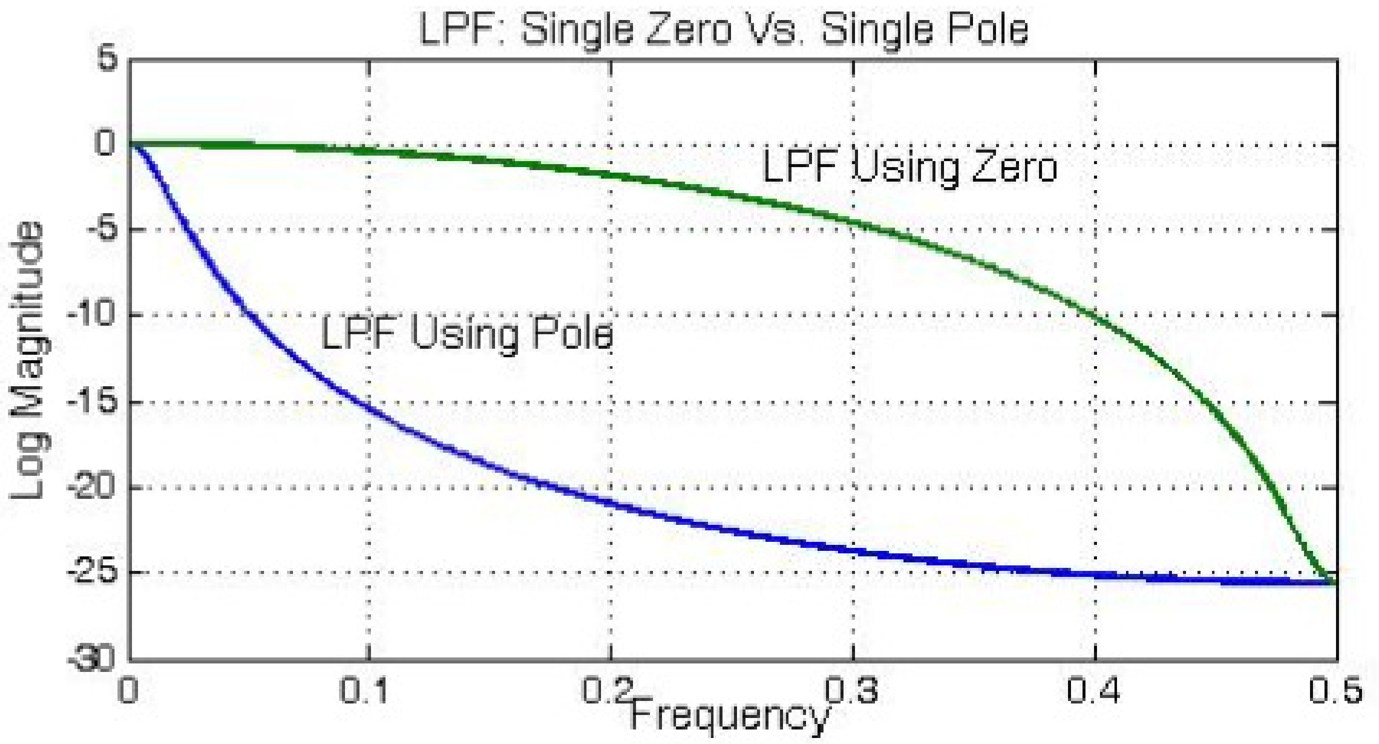
$$= 1.81 \ (2.58 dB)$$

Thus, the 3-dB Bandwidth for an LPF realized using a single zero is $\frac{\pi}{2}$ if $a = 0 \cdot 9$.

The 3-dB BW for the LPF realized using a single pole can be shown to be $\frac{\pi}{30}$, i.e., fifteen times narrower.

Exercise: Derive the 3-dB bandwidth of $H(z) = \frac{1}{1 - az^{-1}}$ $\quad -1 < a < 1$

Poles are more powerful in shaping the frequency response than zeros.

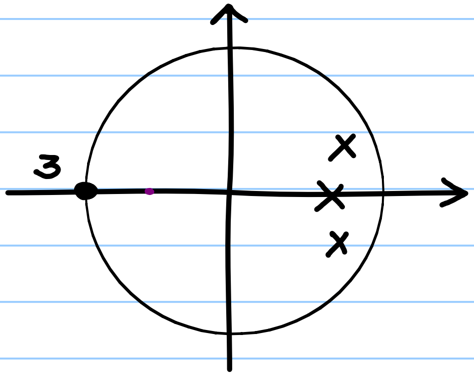


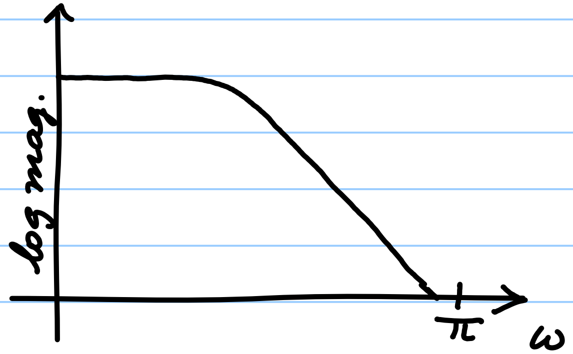
3dB Bandwidth of LPF realized using pole is 15 times smaller!

$H_{LPF}(z) = \frac{1}{1-0.9z^{-1}}$ Vs. $H_{LPF}(z) = 1+0.9z^{-1}$

Peak gain has been normalized to unity.

One can add more poles to get a flatter passband:





Systematic procedures for designing filters will be taught in the Digital Filter Design course.

Classical Analog Filters: Butterworth, Chebychev, Elliptic, Bessel

EE5330 Sep. 18, 2013

Assigning poles close to the unit circle to get a sharp filter makes the response sensitive to pole location.

Consider $f(x_0 + \Delta x) \simeq f(x_0) + \Delta x \cdot f'(x_0)$

$$\Rightarrow \quad f(x_0+\Delta x) - f(x_0) \simeq \Delta x \cdot f'(x_0)$$

$$\Rightarrow \quad \Delta f \simeq \underbrace{\frac{\Delta x}{x_0}}_{\text{relative change in } x_0} \; x_0 f'(x_0)$$

Δf can become large if $f'(x_0)$ is large. Hence, if $f(\cdot)$ represents the frequency response of a system, $f'(\cdot)$ will be large in the transition region of sharp filters. Such responses are sensitive to small changes in pole locations.

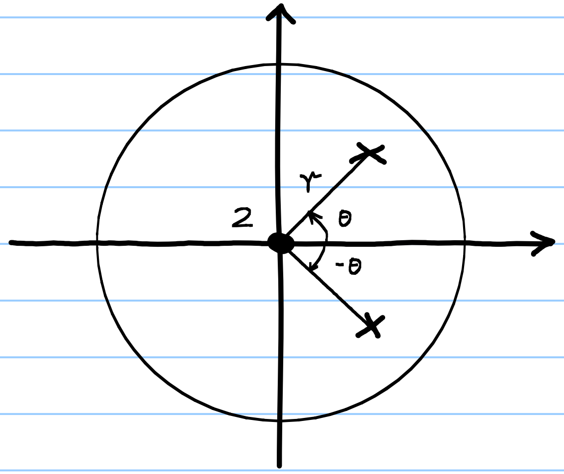
2nd Order Filter:

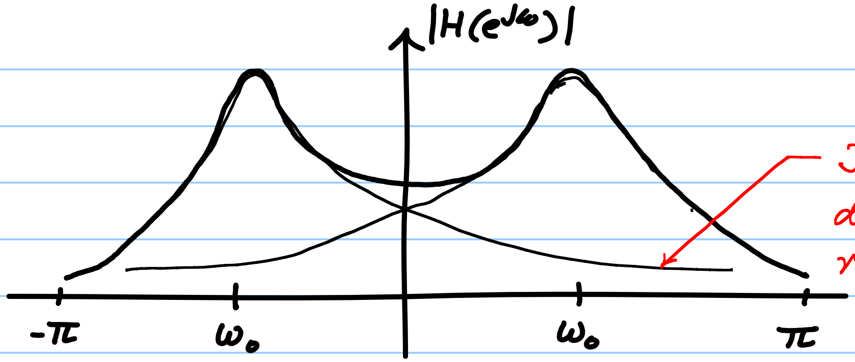
This filter is also called as a RESONATOR

$$h[n] = r^{n} \frac{\sin[\theta(n+1)]}{\sin\theta} u[n]$$

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})} = \frac{1}{1-2r\cos\theta\, z^{-1}+r^{2}z^{-2}}$$

$$|H(e^{j\omega})|^{2} = \frac{1}{[1+r^{2}-2r\cos(\omega-\theta)][1+r^{2}-2r\cos(\omega+\theta)]}$$





Tail of the response due to the pole at $re^{-j\theta}$ will interfere and cause a SHIFT in the peak location! In general, the peak will NOT be at $\omega = \pm\theta$.

The expression for the peak location is obtained by solving

$$\omega_0 = \underset{\omega}{\operatorname{argmax}} \frac{1}{[1+r^2-2r\,\mathrm{Cos}(\omega-\theta)][1+r^2-2r\,\mathrm{Cos}(\omega+\theta)]}$$

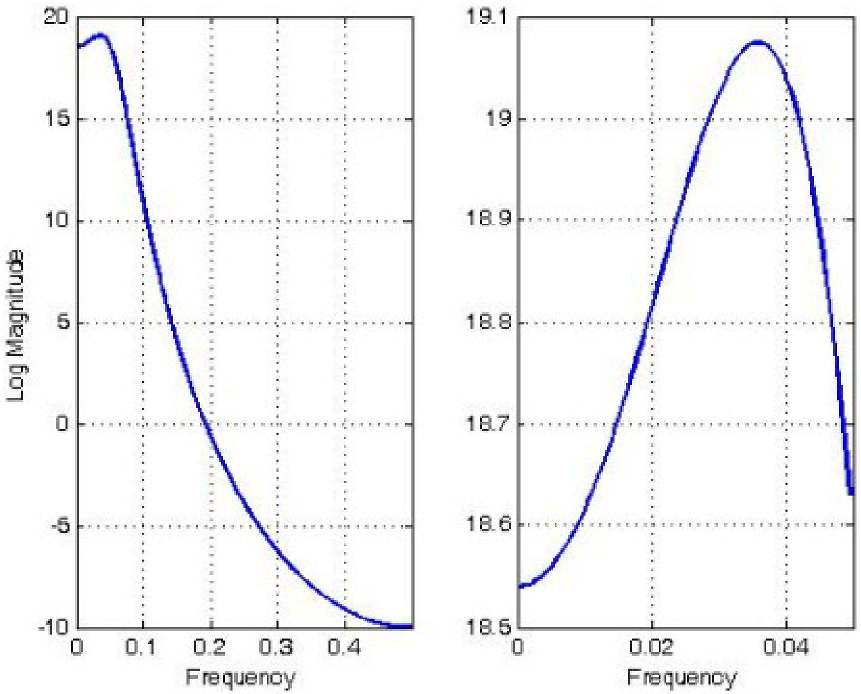
<u>Exercise</u>: Show that $\omega_0 = \mathrm{Cos}^{-1}\left[\frac{1+r^2}{2r}\,\mathrm{Cos}\,\theta\right]$

Interference is reduced if the poles move farther apart. The farthest they can be is when $\theta = \frac{\pi}{2}$. For this value of θ, $\omega_0 = \frac{\pi}{2}$, i.e., there is no shift in peak location for any 'r'!

Interference also reduces as $r \to 1$.

Note that for a distinct peak to be seen at $\omega = \omega_0 \neq 0$, we require $-1 \leq \frac{1+r^2}{2r} \cos\theta \leq 1$

Example of Peak Shifting:



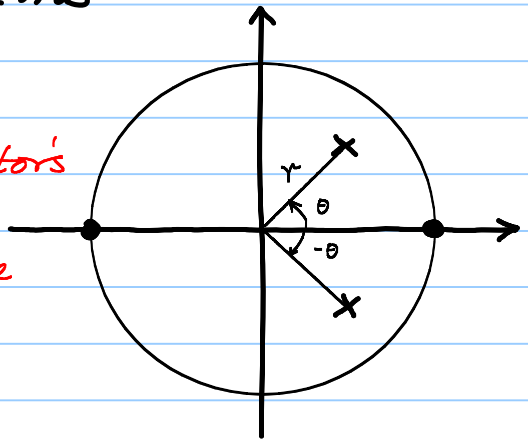
The plots shown on the right correspond to $r = 0.8$ and $\theta = \frac{\pi}{10}$. If there were no interference, the peak would have been at $\omega_0 = \frac{\theta}{2\pi} = 0{\cdot}05$. However, due to interference, the actual peak occurs at $\frac{\omega_0}{2\pi} = 0{\cdot}0358$, as can be seen from the plots.

Improved Resonator :

The resonator is a crude Bandpass Filter. A canonic BPF must completely reject frequency components at $\omega = 0$ and $\omega = \pi$. The given resonator can be modified to reject these two frequency components by adding zeros at $z = \pm 1$

$$H(z) = \frac{(1+z^{-1})(1-z^{-1})}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$

Improved resonator's pole-zero plot. Note the zeros are now at $z = \pm 1$



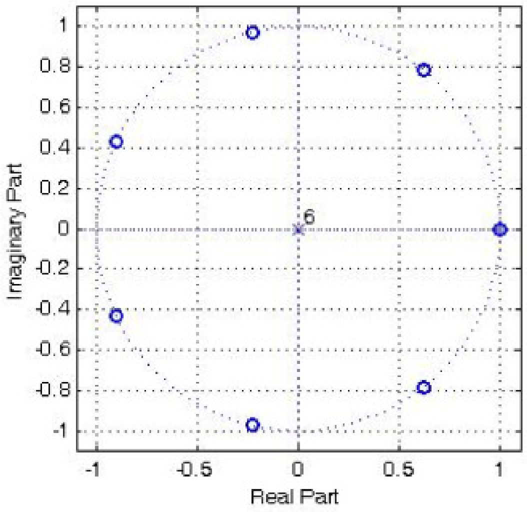
The improved resonator also suffers from peak shifting due to tail interference.

Exercise Derive the expression for the peak location. Comment on the result.

Moving Average Filter:

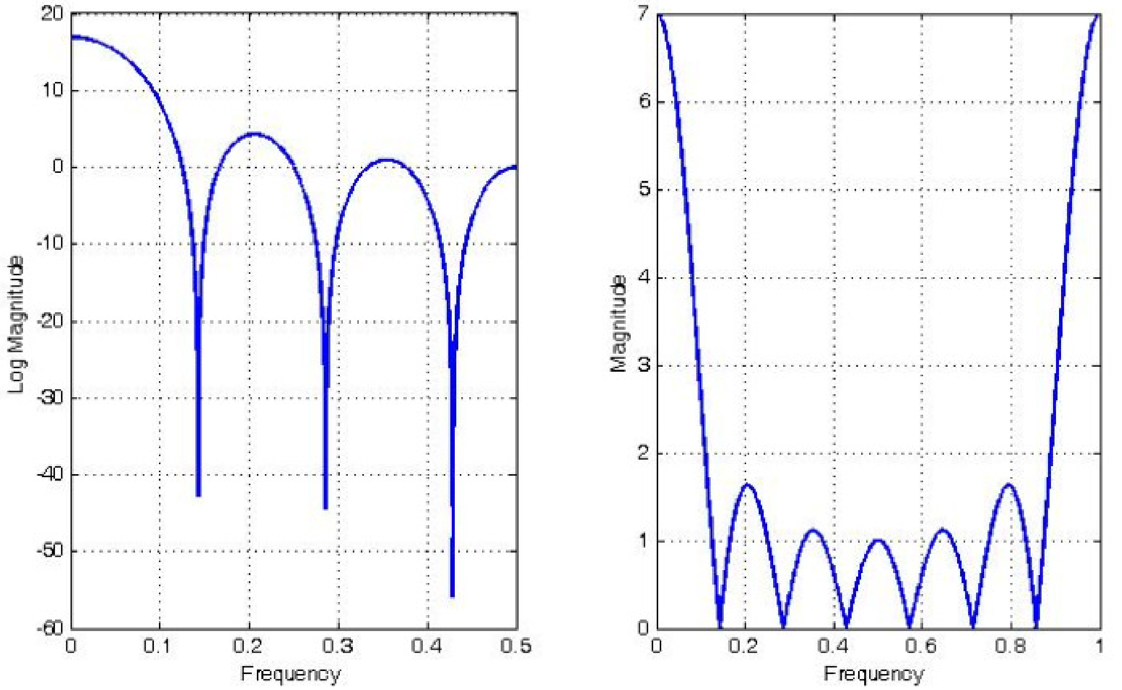
$$h[n] = \frac{1}{N} \quad 0 \le n \le N-1 \quad \longleftrightarrow \quad H(z) = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}}$$

$$H(e^{j\omega}) = \frac{e^{-j\omega N/2}}{N} \frac{\sin N\omega/2}{\sin \omega/2}$$



Pole-Zero Plot of Moving Average Filter

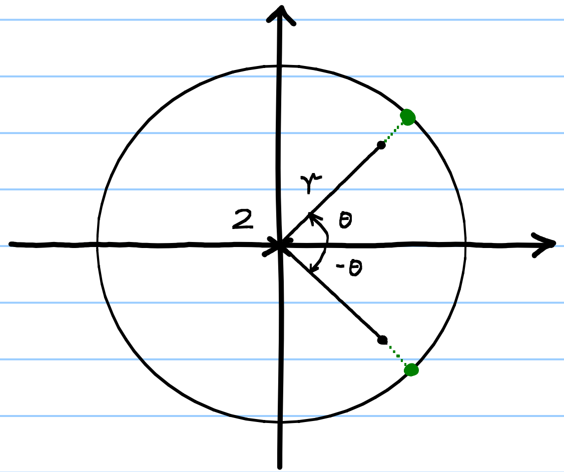
Magnitude Frequency Response shown in both log and linear scales

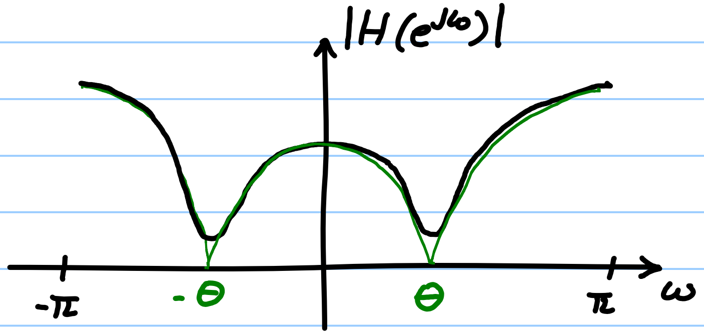


EE5330 Sep. 19, 2013

Notch Filter Used for removing one or more sinusoids.

Consider $H(z) = 1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}$

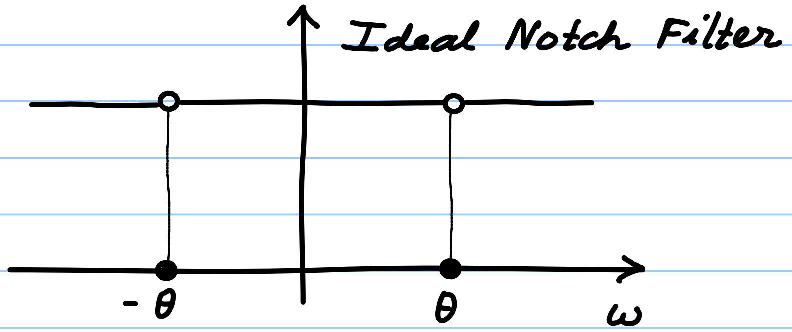




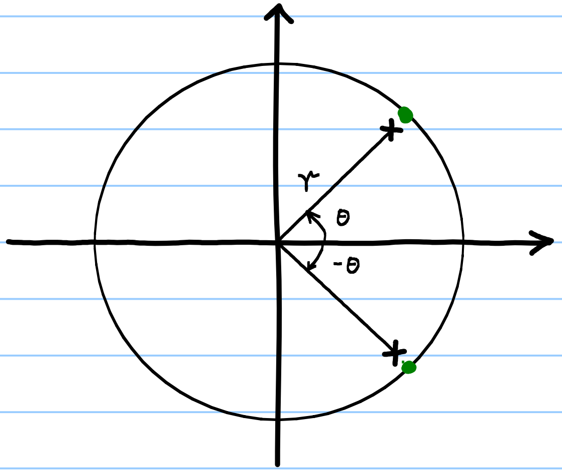
If $r = 1$, the zeros lie on the unit circle and the frequency response goes to zero at $\omega = \pm\theta$ [Green curve in the above fig.]

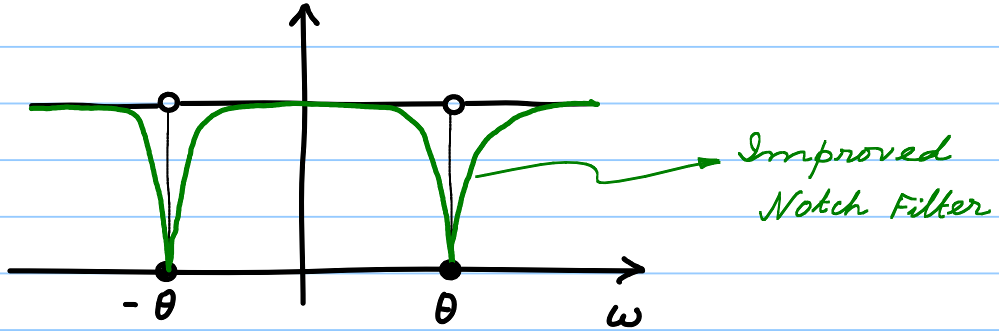
$H(z) = 1 - 2\cos\theta\, z^{-1} + z^{-2}$

While the frequency component at $\omega = \pm\theta$ is nulled, the notch filter's response is far from the ideal response shown below:



The given notch filter's response can be improved by adding poles at $re^{\pm j\theta}$ where r is close to 1.





Improved notch filter: $H(z) = \dfrac{1 - 2\cos\theta\, z^{-1} + z^{-2}}{1 - 2r\cos\theta\, z^{-1} + r^{2} z^{-2}}$

In practice r cannot be made too close to 1 because of limitations imposed by finite precision effects.

Comb Filters

Consider the simple LPF given by $H(z) = \frac{1+z^{-1}}{2}$ ← ensures unity gain at $\omega = 0$

Then,

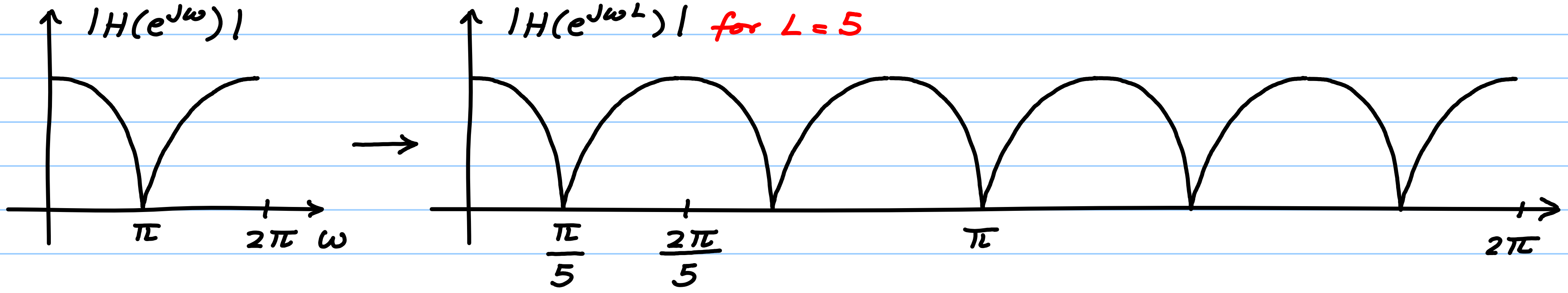
$$H(z^L) = \frac{1+z^{-L}}{2}$$

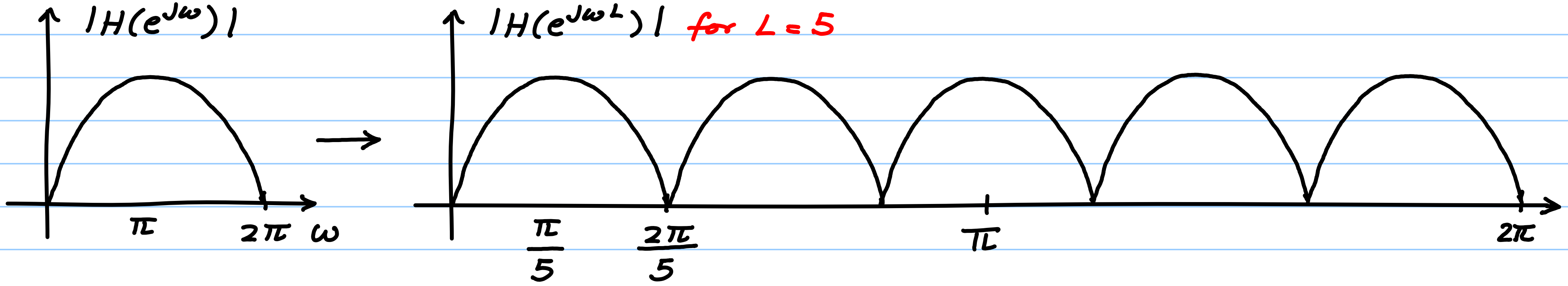
The roots are now the L^{th} roots of -1, i.e., $e^{j(2k+1)\pi/L}$

The peaks occur at $\omega = \frac{2\pi k}{L}$

Similarly, for a highpass filter, we start with $H(z) = \frac{1-z^{-1}}{2}$

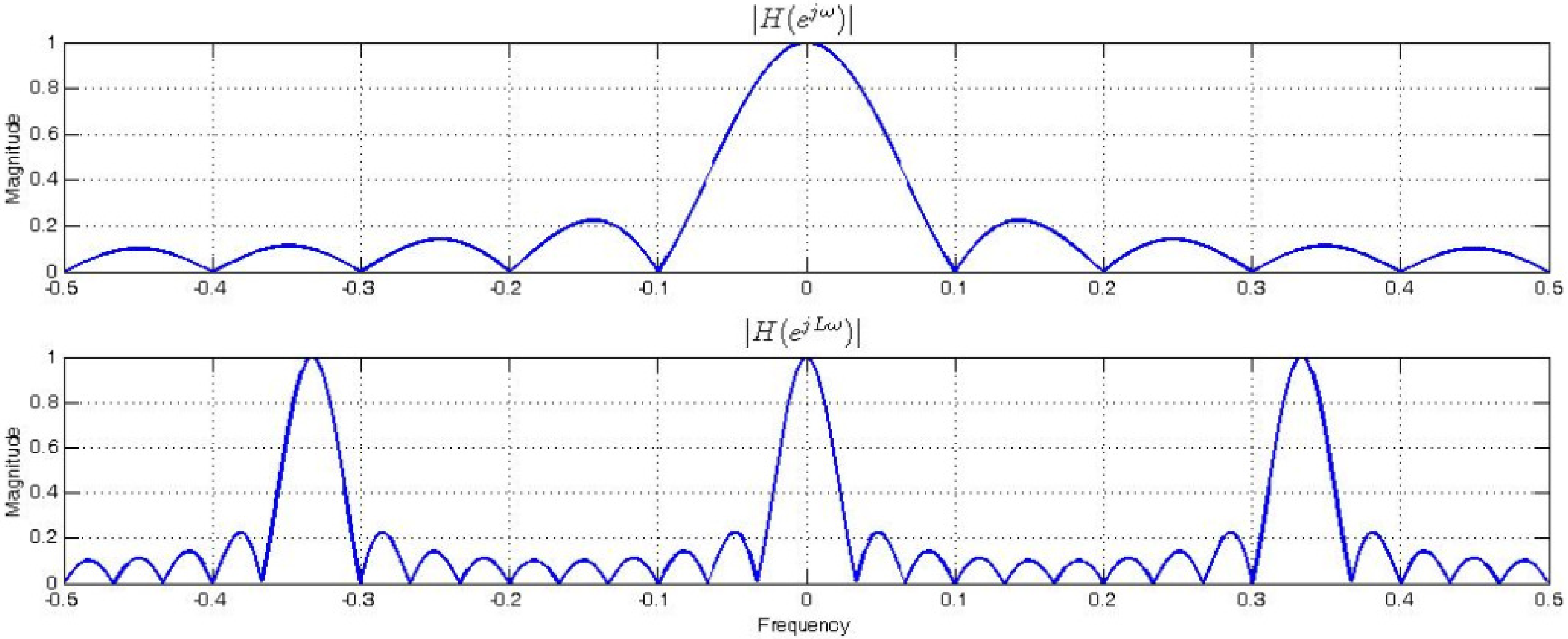
and get $H(z) = \frac{1-z^{-L}}{2}$. Roots: L^{th} roots of 1


$|H(e^{j\omega})|$
π
2π
ω
$|H(e^{j\omega L})|$ for $L=5$
$\frac{\pi}{5}$
$\frac{2\pi}{5}$
π
2π


$|H(e^{j\omega})|$
π
2π
ω
$|H(e^{j\omega L})|$ for $L=5$
$\frac{\pi}{5}$
$\frac{2\pi}{5}$
π
2π

If $H(z) = \frac{1}{N} \frac{1-z^{-N}}{1-z^{-1}}$, then $|H(e^{j\omega})| = \left| \frac{\sin N\omega/2}{\sin \omega/2} \right|$

The plots for N=10 and L=3 are given below:



Phase Response

Recall that the standard form of $H(z)$ for a rational TF is (transfer function)

$$H(z) = b_0 \frac{\prod_{l=1}^{M} (1 - z_l z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

Hence,

$$\measuredangle H(e^{j\omega}) = \arg\{b_0\} + \sum_{l=1}^{M} \arg\{1 - z_l e^{-j\omega}\} - \sum_{k=1}^{N} \arg\{1 - p_k e^{-j\omega}\}$$

The form of a typical term is $\arg\{1 - re^{j\theta} e^{-j\omega}\}$

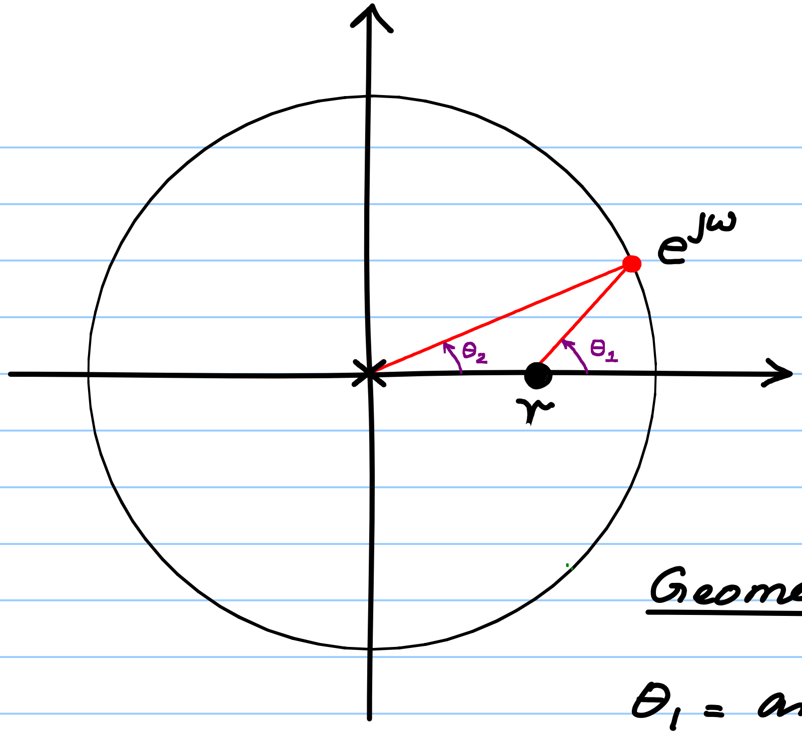
$$\arg\{1 - re^{j\theta}e^{-jw}\} = \arg\{1 - r\cos(\omega-\theta) + j\sin(\omega-\theta)\}$$

$$= \tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

Note: $\tan^{-1}\left(\frac{3}{4}\right) \neq \tan^{-1}\left(\frac{-3}{-4}\right)$

We are interested in the OVERALL phase response. For a system with real-valued coefficients, the phase response will be an ODD function of ω.

Consider the case $0 < r < 1$ and $\theta = 0$, i.e., real-valued impulse response.



$$H(z) = 1 - r z^{-1}$$

$$\measuredangle H(e^{j\omega}) = \arg\{1 - re^{-j\omega}\}$$

$$= \arg\{e^{j\omega} - r\} - \arg\{e^{j\omega}\}$$

Geometric Interpretation:

θ_1 = angle of the vector joining $e^{j\omega}$ & r

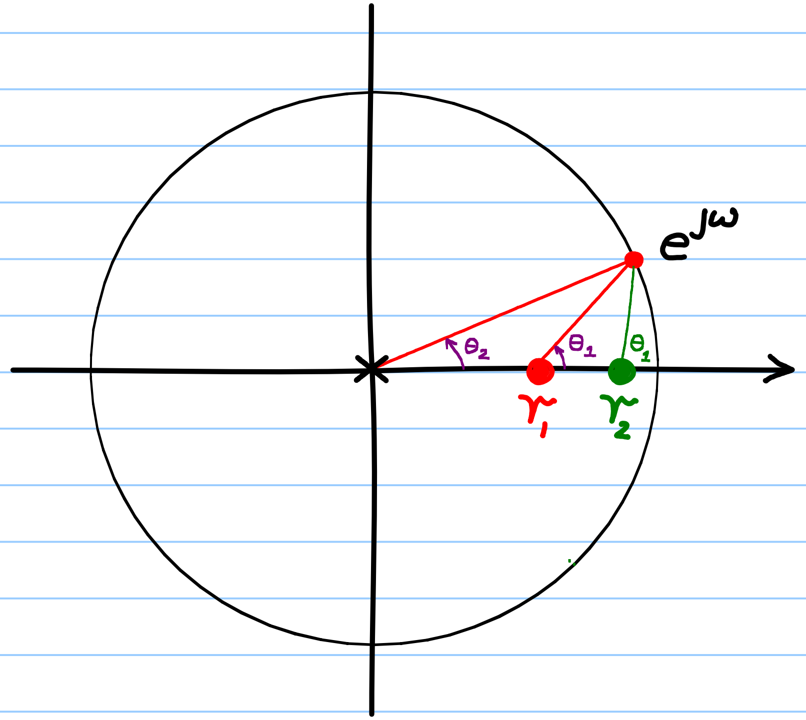
θ_2 = angle of the vector joining $e^{j\omega}$ & 0

Overall angle, i.e., phase response is

$\theta_1 - \theta_2$

Note: Trivial pole or zero i.e., pole or zero at the origin will contribute to the phase response.

EE5330 Sep. 24, 2013



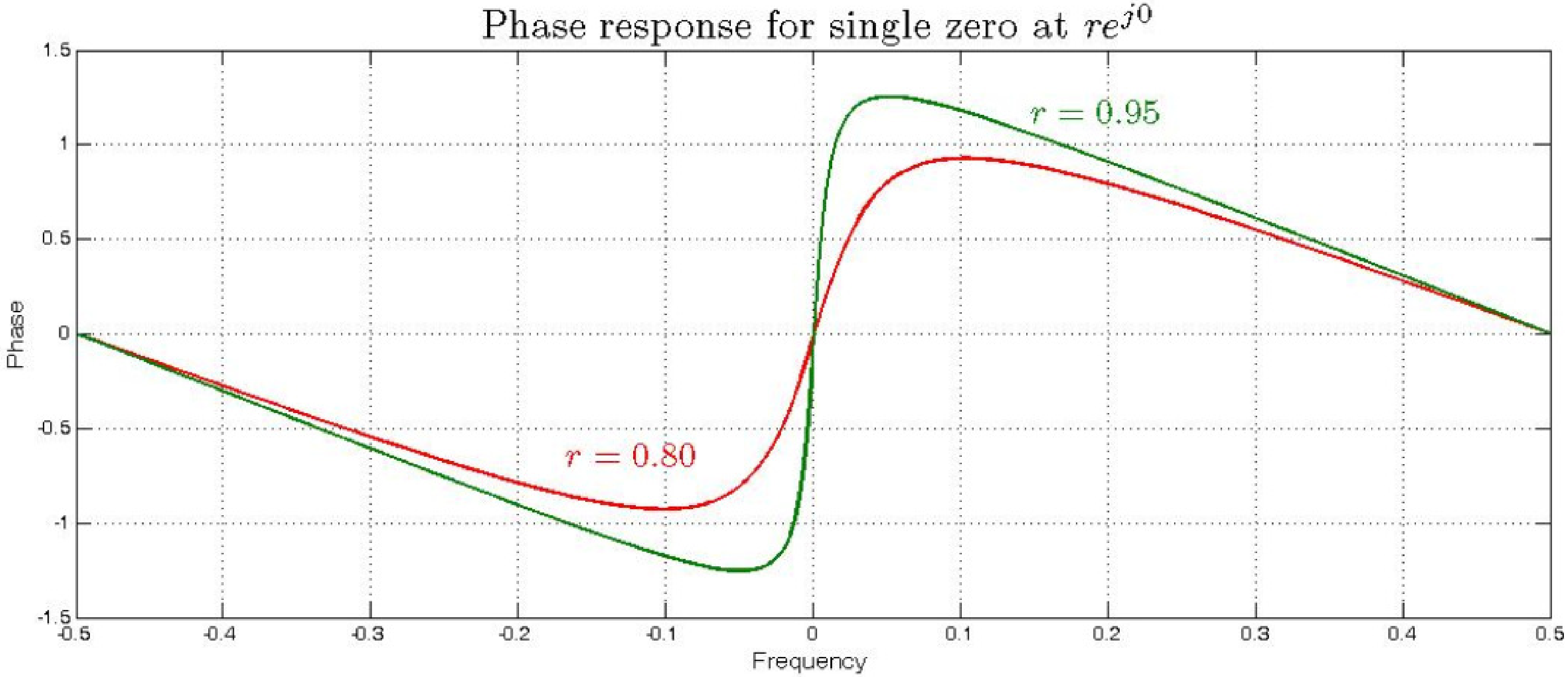
For $0 < r_1 < r_2 < 1$ the shape of the phase response is shown below.

At $\omega = 0$, $\theta_1 = \theta_2 = 0 \Rightarrow \theta_1 - \theta_2 = 0$

At $\omega = \pi$, $\theta_1 = \theta_2 = \pi \Rightarrow \theta_1 - \theta_2 = 0$

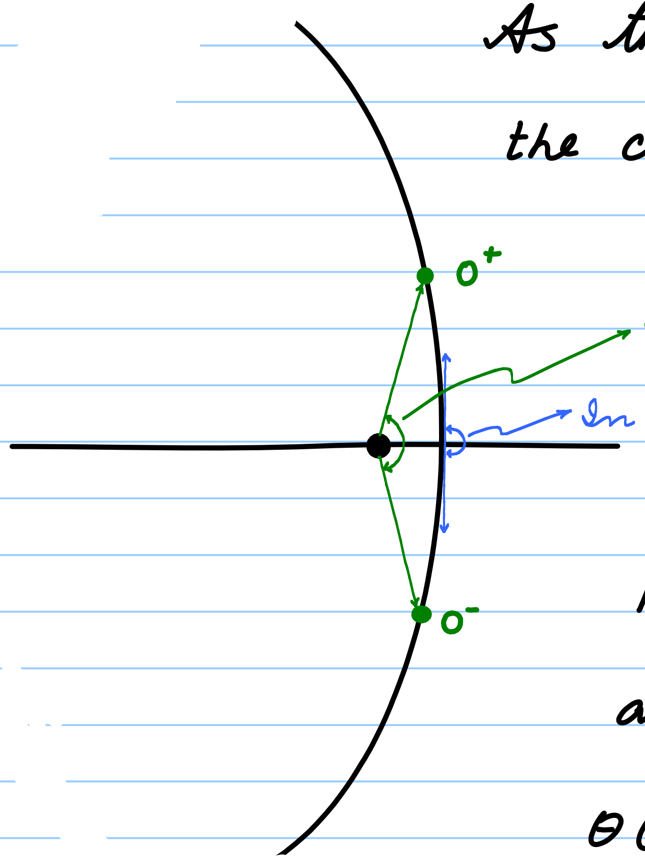
Just beyond $\omega = 0$, θ_1 increases more rapidly than $\theta_2 \Rightarrow \theta_1 - \theta_2 > 0$.

Hence the final shape of the phase response is as follows:



If $\theta \neq 0$, the curve will be centred at $\omega = \theta$ rather than at 0.

As the zero tends towards the unit circle, the change in angle tends towards π :



As r tends to 1, the change in angle tends to π

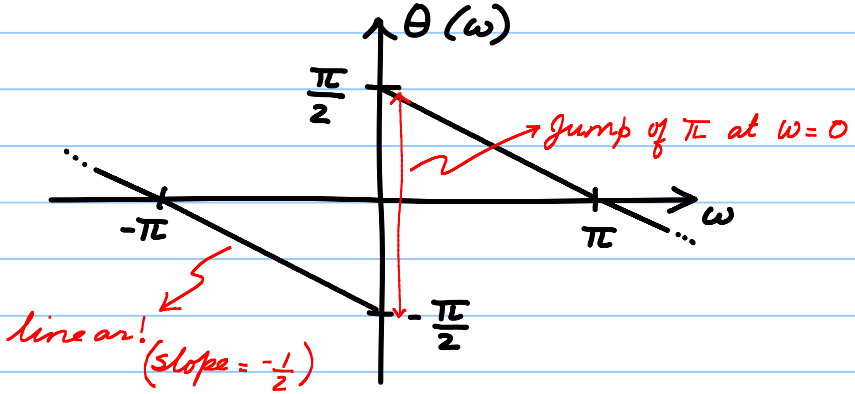
In the limit as r → 1 the change in angle equals $\boxed{\pi}$

For r = 1, the expression for the phase angle becomes

$$\theta(\omega) = \tan^{-1} \frac{\sin \omega}{1 - \cos \omega} = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\omega}{2}\right)\right]$$

Hence,

$$\theta(\omega) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & \omega > 0 \\ -\frac{\pi}{2} - \frac{\omega}{2} & \omega < 0 \end{cases}$$



Important Features:

(a) Phase jump of π at $\omega = 0$

(b) Phase is linear

(c) Slope of the linear phase part is $-\frac{1}{2}$

If $r = 1$ and $\theta \neq 0$, jump of π will occur at $\omega = \theta$

Slope will still be linear with value unchanged from $-\frac{1}{2}$

Any collection of zeros on the unit circle will give rise to an overall phase response that is LINEAR with jumps of π occuring at the locations of the zeros. The slope of the linear region equals $-\frac{N}{2}$. If these zeros occur in complex conjugate pairs, the overall response will be odd symmetric.

Let there be a zero at $\omega = \theta$ on the unit circle.

Let the frequency response be equal to H_1 at $\omega = \theta^-$

Consider the frequency response at $\omega = \theta^+$; let it be H_2.

The change in $H(e^{j\omega})$ due to all the other poles and zeros will be negligible because of the negligible change in both the distances as well as angles. The only change will be contributed by the zero at $\omega = \theta$. This zero contributes a phase change of π. Hence, $H_2 = H_1 \cdot e^{j\pi} = -H_1$

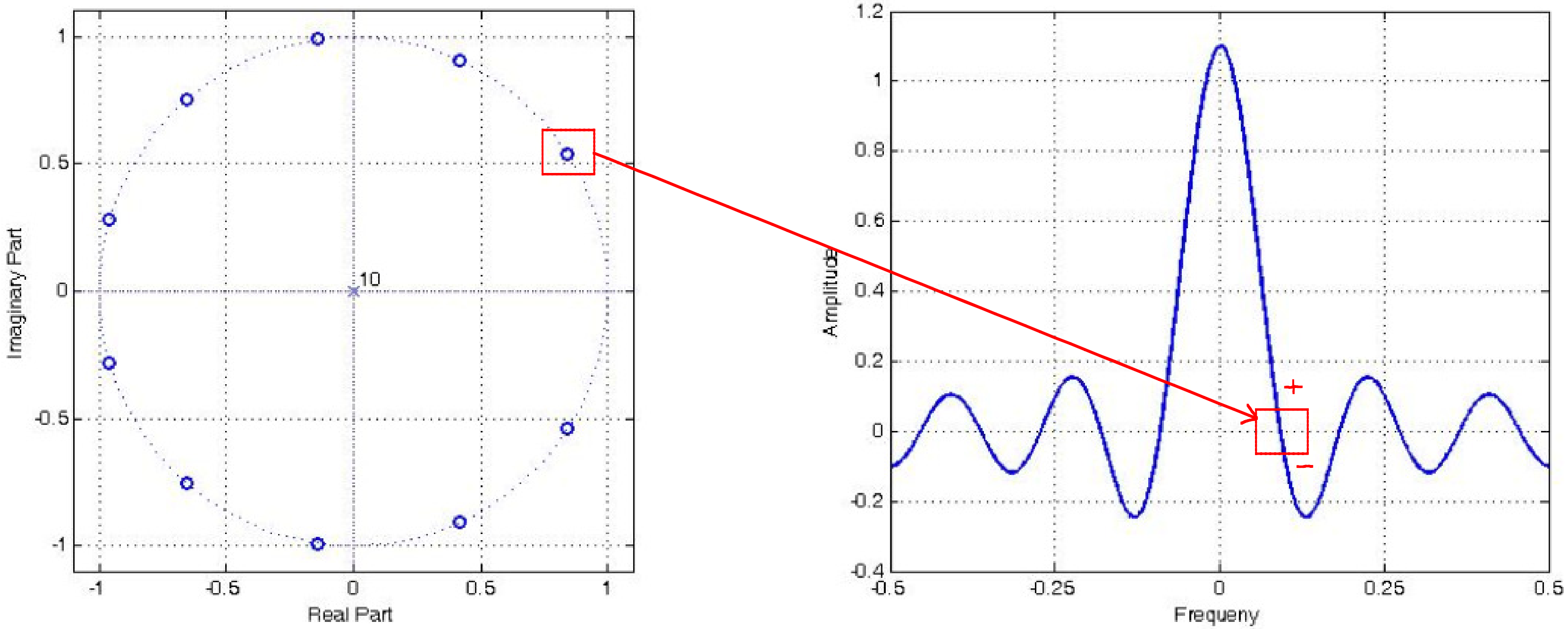
Hence, crossing a first order zero on the unit circle causes a sign change in the frequency response.

Example

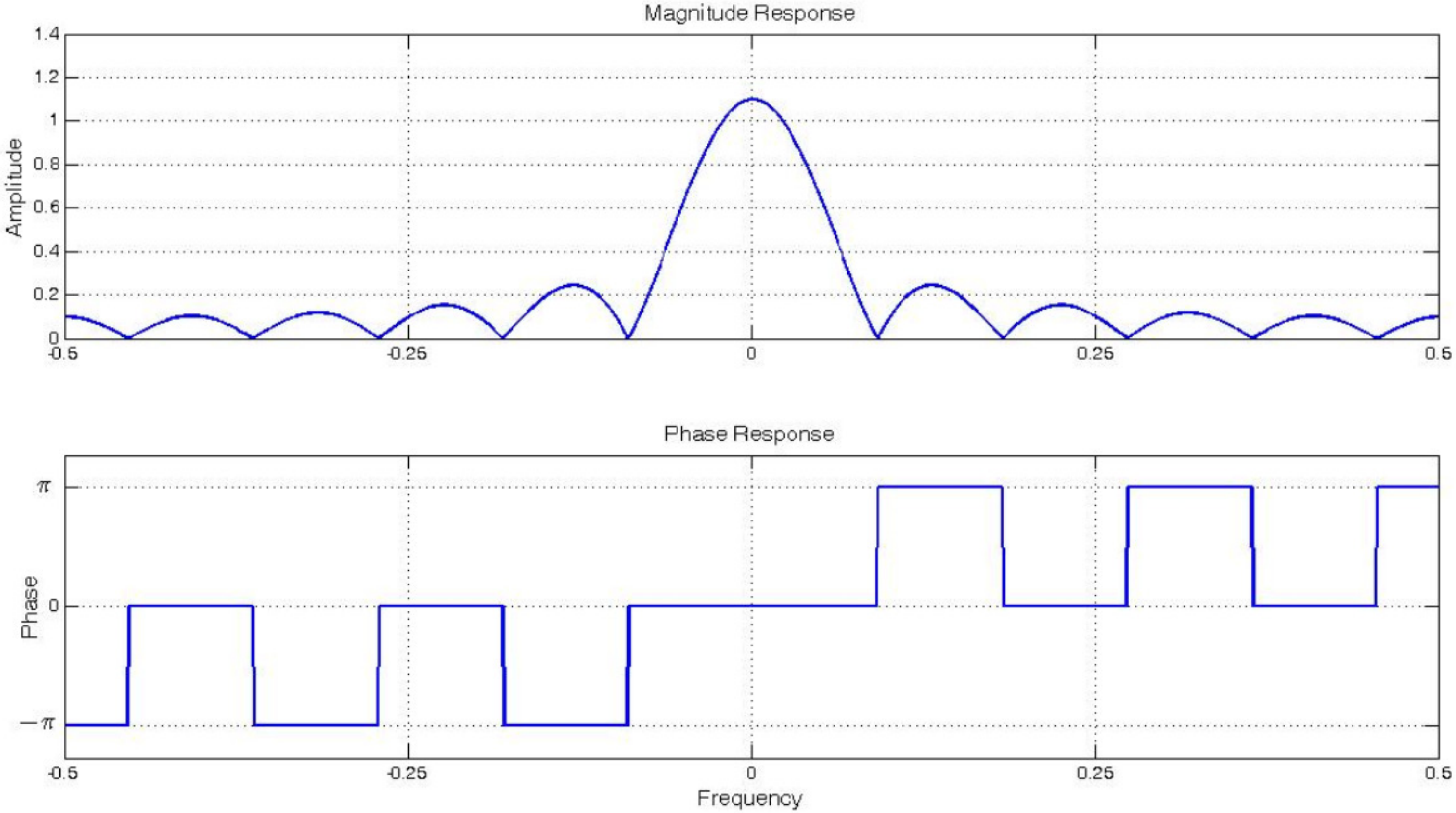
$$h[n] = 1 \qquad -N \leq n \leq N$$

$$H(e^{j\omega}) = \frac{\sin (2N+1)\omega/2}{\sin \omega/2}$$

Crossing each zero introduces a sign change!



If the above frequency response is plotted as two separate plots, i.e., as magnitude and phase plots, the plots will be as shown below:

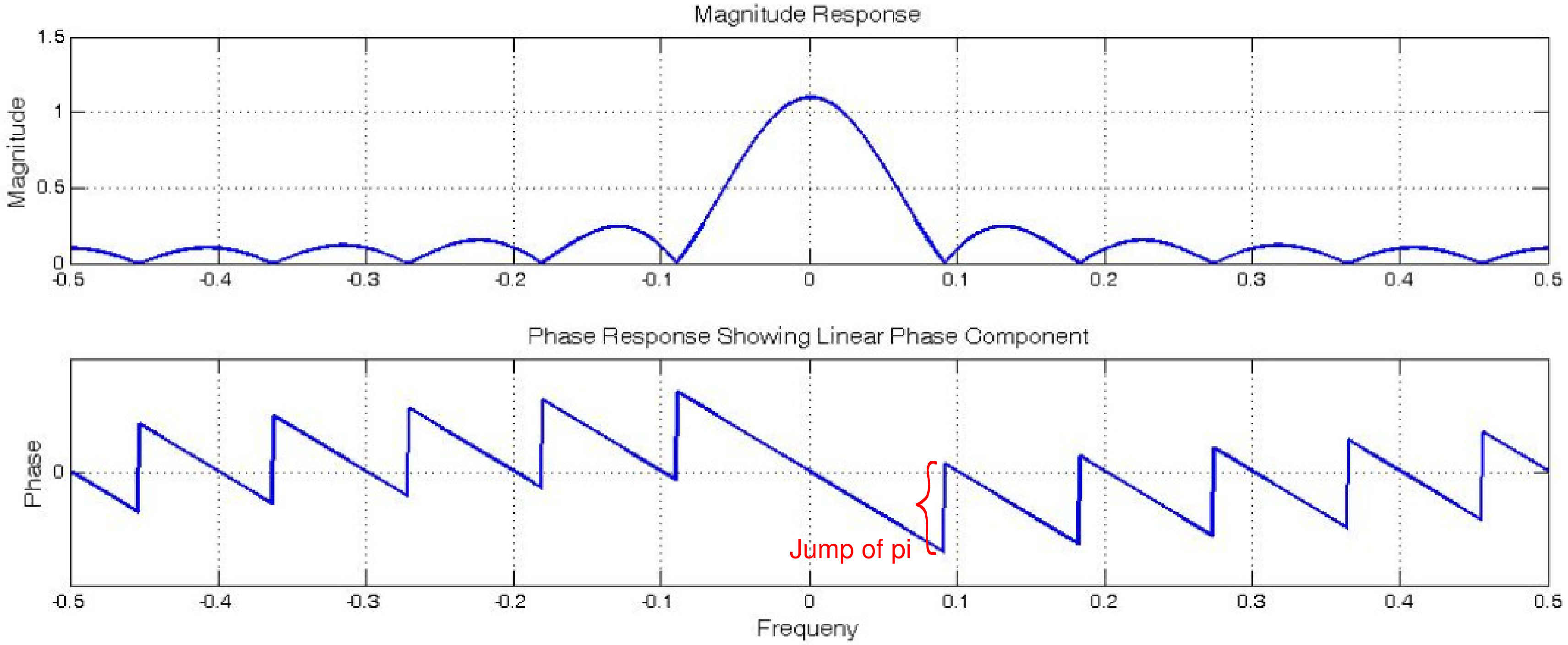
Magnitude Response
Amplitude
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.5
-0.25
0
0.25
0.5
Phase Response
Phase
π
0
−π
-0.5
-0.25
0
0.25
0.5
Frequency

By convention, a sign change is shown as a phase change of π (rather than $-\pi$) for $\omega > 0$.

If $h[n] = 1 \quad 0 \leq n \leq 2N$, $H(e^{j\omega}) = e^{-j\omega N/2} \dfrac{\sin(2N+1)\omega/2}{\sin \omega/2}$

The magnitude plot remains unchanged.

The phase plot acquires a linear phase term with slope equals $-\frac{N}{2}$. The new magnitude and phase plots are shown below.


Magnitude Response
Magnitude
1.5
1
0.5
0
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
Phase Response Showing Linear Phase Component
Phase
0
Jump of pi
Frequeny

When we cross a first order zero on the unit circle, we acquire a phase change of π. If we cross an N^{th} order zero, we acquire a phase change of $N\pi$.

Hence, crossing a 2^{nd} order zero causes a phase change of 2π, which causes no sign change!

Exercise:

Plot the magnitude and phase plots of $g[n] = h[n] * h[n]$ where $h[n]$ is as shown before. Examine behaviour around zero crossings. What is the slope of the frequency response at the zero locations?

EE5330 Sep. 25, 2013

Can there be discontinuities in the phase response other than π ?

Recall that

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)} = H_R(e^{j\omega}) + j\, H_I(e^{j\omega})$$

$$|H(e^{j\omega})|^2 = H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})$$

$$\theta(\omega) = \begin{cases} \tan^{-1} \dfrac{H_I(e^{j\omega})}{H_R(e^{j\omega})} & H(e^{j\omega}) \neq 0 \\ \text{undefined} & H(e^{j\omega}) = 0 \end{cases}$$

By definition, $\theta(\omega) \in (-\pi, \pi]$ i.e., $-\pi < \theta(\omega) \leq \pi$

Note that $e^{j\omega} \in$ RoC of $H(z)$ and $H(z)$ is a rational transfer function

Hence $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are continuous functions of ω.

Discontinuities in $\theta(\omega)$ will occur in the following two cases:

(i) At points where $H_I(e^{j\omega_0}) = 0$ and $H_R(e^{j\omega_0}) < 0$, $\theta(\omega_0) = \pi$.

If $H_I(e^{j\omega_0^-}) < 0$ or $H_I(e^{j\omega_0^+}) < 0$ (or both), then

$\theta(\omega_0^-) = -\pi$, $\theta(\omega_0^+) = -\pi$. Hence, the phase jumps by 2π,

i.e., $\pi - (-\pi) = 2\pi$.

(ii) If $H(e^{j\omega_0}) = 0$, then $\theta(\omega_0)$ is undefined and hence phase cannot be continuous at that point.

The no. of points at which the phase can become discontinuous is finite because $H(z)$ is rational.

Jumps of 2π in $\theta(\omega)$ can be removed by adding or subtracting integer multiples of 2π - called $\boxed{\text{PHASE UNWRAPPING}}$

If we define $\theta(\omega)$ suitably at points where $H(e^{j\omega}) = 0$, is it possible to get rid of discontinuities in $\theta(\omega)$?

The answer is NO because we cannot get rid of jumps of π (odd multiples) by phase unwrapping.

Nevertheless, there is a way to make the phase continuous for systems with rational transfer functions.

Crossing a zero on the unit circle introduces a sign change. But $|H(e^{j\omega})|$ is constrained to be non-negative. Hence the phase is forced to jump by π.

If we replace $|H(e^{j\omega})|$ by $A(\omega)$, where $A(\omega) \in \mathbb{R}$, then the change of sign can be absorbed in $A(\omega)$ and the phase can remain continuous.

Hence, we decompose the frequency response as

$$H(e^{j\omega}) = \underbrace{A(\omega)}_{\substack{\text{real-valued, i.e.,} \\ \text{can take on both} \\ \text{+ve and -ve values.}}} e^{j\underbrace{\phi(\omega)}_{\boxed{\text{continuous}}\ \substack{\text{phase} \\ \text{function}}}}$$

The decomposition $A(\omega)e^{j\phi(\omega)}$ is not unique because $A(\omega)\,e^{j\phi(\omega)}$ is the same as $-A(\omega)\,e^{j(\phi(\omega)+\pi)}$.

The decomposition can be made unique if we enforce the following constraint:

$$0 \leq \phi(0) < \pi$$

Example

Recall the example where $h[n] = 1 \quad 0 \leq n \leq 2N$

$$H(e^{j\omega}) = e^{-j\omega N} \frac{\sin (2N+1)\omega/2}{\sin \omega/2}$$

The usual magnitude-phase decomposition resulted in a $\theta(\omega)$ that had jumps of π at the zero crossings. If we replace $|H(e^{j\omega})|$ by $A(\omega)$ where

$$A(\omega) = \frac{\sin (2N+1)\, \omega/2}{\sin \omega/2}$$

then $\phi(\omega) = -N\omega$, which is now continuous. Note that $A(\omega)$ given above now takes on both +ve and -ve values.

Phase response of a single complex pole is the negative of the phase response of a single complex zero.

Overall response is the result of the responses due to the individual poles and zeros.

Take a look at Example 5.10 in Oppenheim and Schafer's, "Discrete-Time Signal Processing" (2nd edition)

See also MATLAB's UNWRAP and ANGLE commands.

When we were discussing causal signals, we saw that $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are not independent but related. Does it mean $|H(e^{j\omega})|$ and $\theta(\omega)$ are also related ?

Consider

$$H_1(z) = 1 - az^{-1} = 1 - re^{j\theta}z^{-1}$$

$$|H_1(e^{j\omega})|^2 = 1 + r^2 - 2r\cos(\omega-\theta)$$

$$\theta_1(\omega) = \tan^{-1}\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$$

Now let $H_2(z) = -a^* + z^{-1} = -re^{-j\theta} + z^{-1}$

$$H_2(e^{j\omega}) = -re^{-j\theta} + e^{-j\omega}$$

$$H_2(e^{j\omega}) \cdot H_2^*(e^{j\omega}) = (e^{-j\omega} - re^{-j\theta})(e^{j\omega} - re^{j\theta})$$

$$= 1 + r^2 - 2r\cos(\omega - \theta)$$

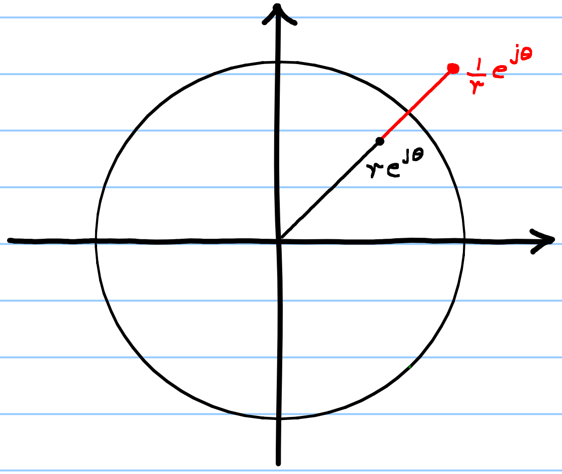
$= |H_1(e^{j\omega})|^2$ same magnitude response!

Phase response is different:

$$\theta_2(\omega) = \tan^{-1} \frac{r\sin\theta - \sin\omega}{\cos\omega - r\cos\theta}$$

Zero of $H_1(z)$ is at $re^{j\theta}$

zero of $H_2(z)$ is at $\frac{1}{r} e^{j\theta}$ i.e., the old zero is reflected about the unit circle.

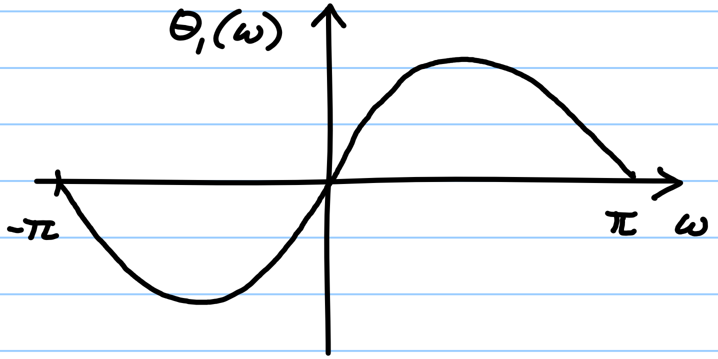


Note :
$$H_2(z) = -a^* + z^{-1}$$
$$= -a^* \left[1 - \frac{1}{a^*} z^{-1} \right]$$
$$= \underbrace{-re^{-j\theta}}_{\text{scale factor needed to make the magnitude identical}} \left[1 - \frac{1}{r} e^{j\theta} z^{-1} \right]$$

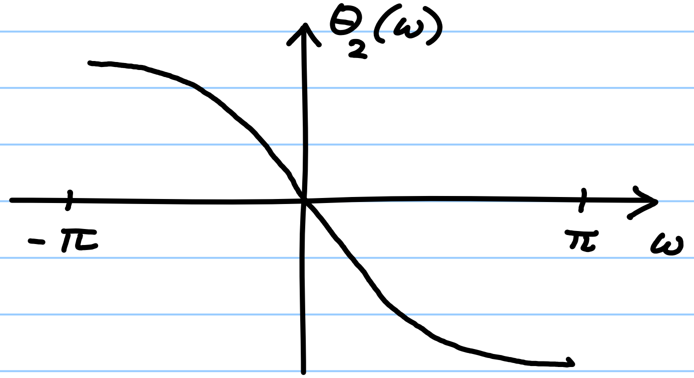
Note Title 26-09-2013

EE 5330 Sep. 26, 2013

$$H_1(z) = 1 - \frac{1}{2} z^{-1}$$



$$H_2(z) = -\frac{1}{2} + z^{-1}$$



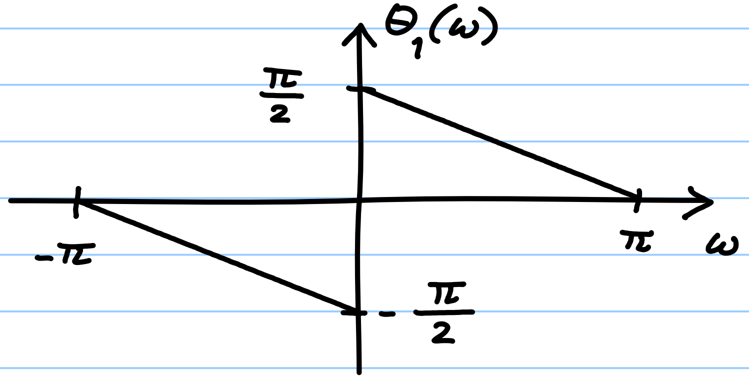
From our previous result,

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$$

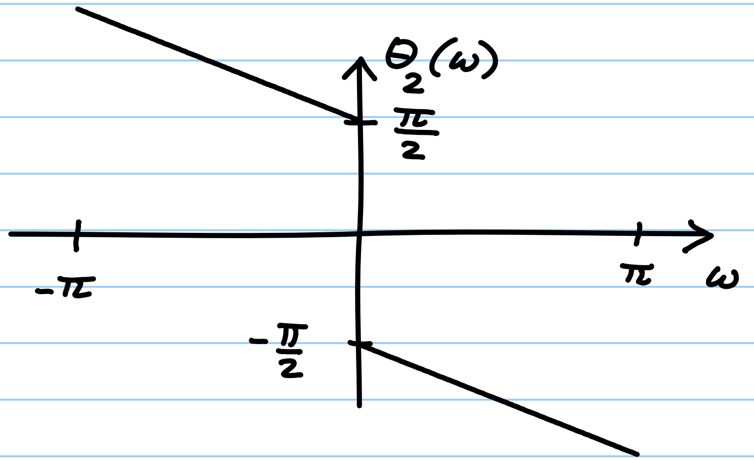
but $\theta_1(\omega)$ and $\theta_2(\omega)$ are very different.

Another example:

$G_1(z) = 1 - z^{-1}$



$G_2(z) = -1 + z^{-1}$



Since $-1+z^{-1} = -(1-z^{-1})$, the multiplication by -1 results in a shift by π in the phase

Since $1-az^{-1}$ and $-a^*+z^{-1}$ have identical magnitude response,

$$H(z) = \frac{-a^* + z^{-1}}{1-az^{-1}}$$

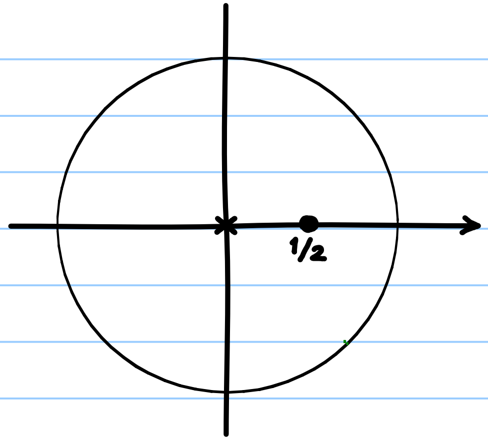
has unit magnitude response. This can also be seen from

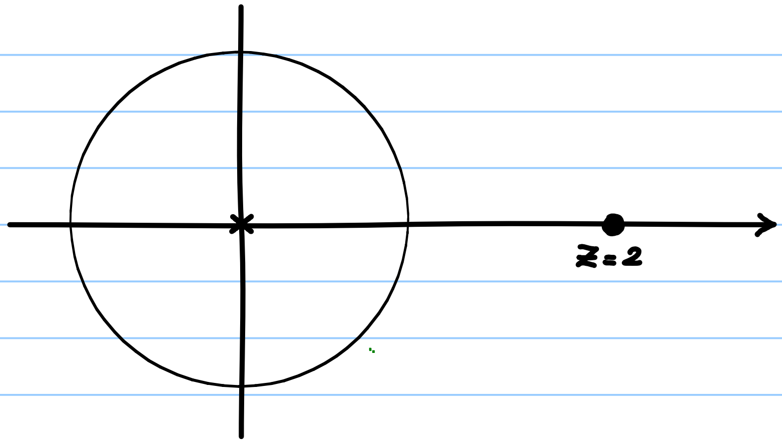
$$H(e^{jw}) = \frac{-a^* + e^{-jw}}{1-ae^{-jw}} = \frac{e^{-jw} - a^*}{e^{-jw}(e^{jw}-a)} \Rightarrow \left| \frac{1}{e^{-jw}} \frac{(e^{jw}-a)^*}{(e^{jw}-a)} \right| = 1$$

A filter with unit or constant magnitude response is called as an $\boxed{\text{ALLPASS Filter}}$.

Note that both $H_1(z)$ and $H_2(z)$ are causal filters. Nevertheless, knowing the magnitude response does not help us in determining the phase response. However, if the filter transfer function is rational, whether or not the system is causal, for a given magnitude response, the number of choices for the phase response is fixed, provided the filter order is specified.

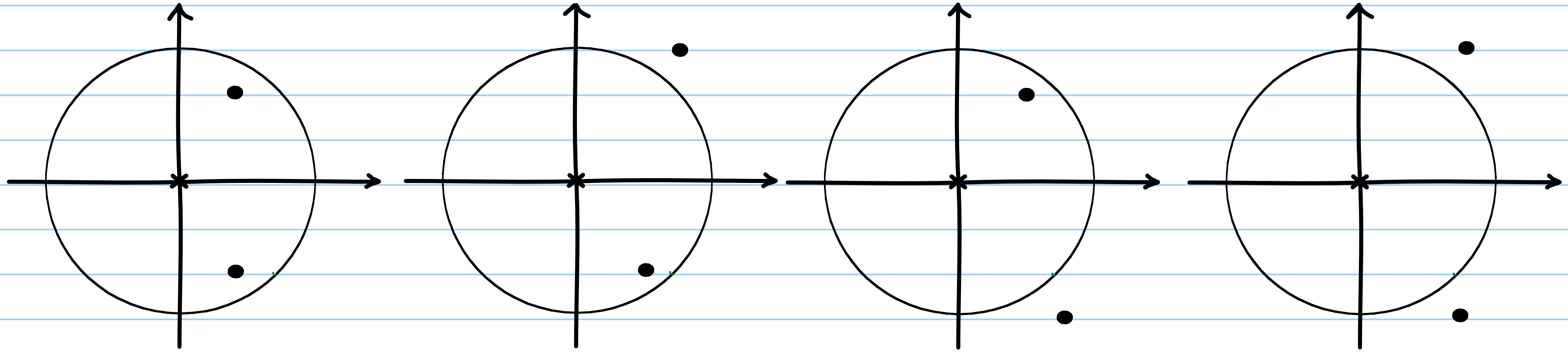
In the case of $1-\frac{1}{2}z^{-1}$, the only other system with identical magnitude response is $-\frac{1}{2}+z^{-1}$.





For $H(z) = 1 - \frac{1}{2} z^{-1}$ there are TWO systems with identical mag. response

For $H(z) = (1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})$, there are FOUR such possibilities:



For a system with N poles and M zeros, can you guess how many possibilities exist ?

In some practical cases, we are given $|H(e^{j\omega})|^2$ and required to find $H(z)$.

Recall that $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$ and $|H(e^{j\omega})|^2 = H(z)H^*(1/z^*)\Big|_{z=e^{j\omega}}$

A general expression for $H(z)$ is $b_0 \dfrac{\prod_{\ell=1}^{M}(1-c_\ell z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$

Assuming $b_0 \in \mathbb{R}$,

$$H^*(1/z^*) = b_0 \frac{\prod_{l=1}^{M}(1 - c_l^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}$$

Hence,

$$C(z) = H(z)\, H^*(1/z^*)$$

$$= b_0^2 \frac{\prod_{l=1}^{M}(1 - c_l z^{-1})(1 - c_l^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

If c_k is a zero of $H(z)$, c_k is also a zero of $C(z)$. In addition $1/c_k^*$ is also a zero of $C(z)$. Similarly, d_k and $1/d_k^*$ are the poles of $C(z)$.

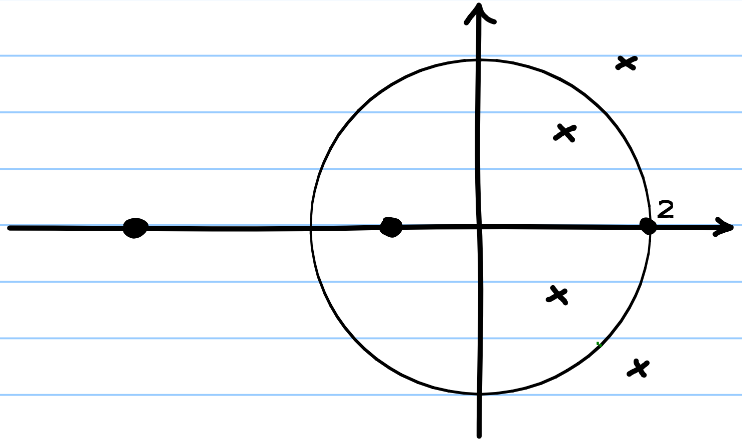
Example

$$H_1(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

$$H_2(z) = \frac{(1-z^{-1})(2+z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

Verify that $H_1(z)H_1^*(1/z^*) = H_2(z)H_2^*(1/z^*) = C(z)$

The pole-zero plot of $C(z)$ is given below:



Problem: We cannot go from $C(z)$ to $H(z)$ in a unique manner. If we assume causal & stable systems, then the poles of $H(z)$ have to be inside the unit circle. But the zeros can be anywhere!

Exercise How many different $H(z)$'s give rise to the given $C(z)$?

Problem: Given $|H(e^{j\omega})|^2$, how can we get $H(z)$ under the condition that $H(z)$ is assumed to be rational?

Let $H(z)$ be of the form $\frac{B(z)}{A(z)}$ where the polynomial coefficients are real-valued. Using the factored form, we saw that

$$H(z)H^*(1/z^*) = b_0^2 \frac{\prod_{l=1}^{M}(1-c_l z^{-1})(1-c_l^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

Since the coefficients are real-valued, the roots occur in conjugate pairs.

Hence the factors related to c_ℓ are:

$$(1-c_\ell z^{-1})(1-c_\ell^* z)(1-c_\ell^* z^{-1})(1-c_\ell z)$$

But, $(1-c_\ell z^{-1})(1-c_\ell z) = 1 - c_\ell(z+z^{-1}) + c_\ell^2$

and $(1-c_\ell^* z^{-1})(1-c_\ell^* z) = 1 - c_\ell^*(z+z^{-1}) + c_\ell^{*2}$

$\Rightarrow$ $H(z)H^*(1/z^*)$ is a function of $z+z^{-1}$

Thus,

$$H(z)H^*(1/z^*) = V(w) \quad \text{where} \quad w = \frac{1}{2}(z+z^{-1})$$

Since $h[n] \in \mathbb{R}$, $H(z) = H^*(z^*)$. Hence $V(w) = H(z)H(1/z)$

Evaluating the above at $z = e^{j\omega}$, we get

$$|H(e^{j\omega})|^2 = V(\cos\omega) = A^2(\omega)$$

Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$H(z)\, H^*(1/z^*) = \frac{10 - 3(z + z^{-1})}{\frac{5}{4} - \frac{1}{2}(z + z^{-1})}$$

$$|H(e^{j\omega})|^2 = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega} = A^2(\omega)$$

Conversely, given $A^2(\omega)$, the steps to get $H(z)$ are:

1) Replace $\cos\omega$ by w to get $V(w)$

2) Find the roots w_i of the num. and den. of $V(w)$.

3) Form the equation $\frac{1}{2}(z+z^{-1}) = w_i$ for each w_i. Let the roots be z_i and $1/z_i$ where z_i denotes the root inside the unit circle.

4) Zeros/Poles of the unknown $H(z)$ are the z_i so obtained.

5) The constant K associated with $H(z)$ is obtained using $H^2(1) = V(1)$

Example

$$A^2(\omega) = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega}$$

$$\mathcal{V}(w) = \frac{10 - 6w}{\frac{5}{4} - w}$$

$$w_1 = \frac{5}{3} \text{ (zero)} \Rightarrow \frac{1}{2}(z + z^{-1}) = \frac{5}{3} \Rightarrow z_1 = \frac{1}{3} \;\&\; \frac{1}{z_1} = 3$$

$$w_2 = \frac{5}{4} \text{ (pole)} \Rightarrow \frac{1}{2}(z + z^{-1}) = \frac{5}{4} \Rightarrow z_2 = \frac{1}{2} \;\&\; \frac{1}{z_2} = 2$$

$$H(z) = K \frac{z - 1/3}{z - 1/2} \qquad H^2(1) = A^2 \left(\frac{2/3}{1/2}\right)^2 = \mathcal{V}(1) = 16 \Rightarrow K = 3$$

Thus, $H(z) = \dfrac{3z-1}{z-\frac{1}{2}}$

By construction, since z_i is the root that is inside the unit circle, $H(z)$ has all its poles and zeros inside the unit circle.

A filter whose poles and zeros are inside the unit circle is called as a **MINIMUM PHASE** filter.

We will see more about min^m phase filters later. The process of getting $H(z)$ from $|H(e^{j\omega})|^2$ is called **Spectral Factorization**.

Alternate Method:

More insight into the spectral factorization problem can be obtained by considering the following alternate approach.

Consider the mapping $Z = \frac{\eta - 1}{\eta + 1}$

(i) For $\eta = e^{j\omega}$, $Z = j \tan \frac{\omega}{2}$ ⇒ as ω goes from $-\pi$ to π, Z goes from $-j\infty$ to $+j\infty$. That is the unit circle is mapped to the imaginary axis.

(ii) Let $|z| \leq 1$. It can easily be seen that this region is mapped to the region $Re\{Z\} \leq 0$, i.e., the region inside the unit circle is mapped to the left-half plane.

(iii) Let $|z| > 1$. It can easily be seen that this region is mapped to the region $Re\{Z\} > 0$, i.e., the region outside the unit circle is mapped to the right-half plane.

EE5330 Oct. 1, 2013

Define $W = \dfrac{w-1}{w+1} \Rightarrow w = \dfrac{1+W}{1-W}$

Since $Z = \dfrac{\eta-1}{\eta+1}$, we get $\eta = \dfrac{1+Z}{1-Z}$

Recall that $w = \frac{1}{2}(\eta + \eta^{-1})$. Hence $\underbrace{\dfrac{w-1}{w+1}}_{W} = \underbrace{\left[\dfrac{\eta-1}{\eta+1}\right]^2}_{Z^2} \Rightarrow W = Z^2$

Therefore, $H(\eta)H(1/\eta) = H\left(\dfrac{1+Z}{1-Z}\right)H\left(\dfrac{1-Z}{1+Z}\right) = V(w) = V\left(\dfrac{1+W}{1-W}\right)$

Define $H\left(\dfrac{1+Z}{1-Z}\right) = H_1(Z) \qquad V_1(W) = V\left(\dfrac{1+W}{1-W}\right)$

Hence, $V_1(W) = H_1(Z)H_1(-Z) = V_1(Z^2)$

The above formulation is analogous to the spectral factorization problem of continuous-time systems with rational transfer function.

The steps for the alternate method are:

1) Replace $\cos w$ by $\dfrac{1+W}{1-W}$ to get $V_1(W)$

2) Find all the roots W_i of $V_1(W)$.

3) Form the eqn $Z^2 = W_i \Rightarrow Z_i = \sqrt{W_i}$ and $-\sqrt{W_i}$

Z_i denotes the root with negative real part.

4) The poles/zeros of $H_1(Z)$ are the Z_i so obtained.

5) The unknown $H(\eta)$ equals $H_1\left(\frac{\eta - 1}{\eta + 1}\right)$.

The gain term is found from $H_1^2(0) = V_1(0)$.

Example

$$A^2(\omega) = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega}$$

$$V_1(W) = \frac{4 - 16W}{\frac{1}{4} - \frac{9}{4}W} \qquad W_1 = \frac{1}{4}, \quad W_2 = \frac{1}{9}$$

$Z_1^2 = \frac{1}{4} \Rightarrow Z_1 = -\frac{1}{2}$ (solution with negative real part)

$Z_2^2 = \frac{1}{9} \Rightarrow Z_2 = -\frac{1}{3}$

$$H_1(Z_1) = K \cdot \frac{Z + \frac{1}{2}}{Z + \frac{1}{3}}$$

$$H_1^2(0) = K^2 (3/2)^2 = \tilde{V}_1(0) = 16 \Rightarrow K = \frac{8}{3}$$

$$H(\eta) = H_1\left(\frac{\eta - 1}{\eta + 1}\right) = \frac{8}{3} \frac{9\eta - 3}{8\eta - 4} = \frac{3\eta - 1}{\eta - \frac{1}{2}}$$

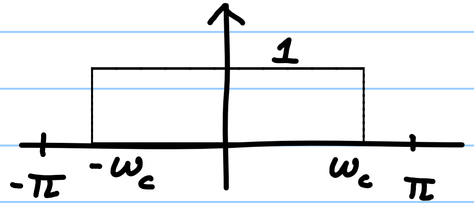
Note: By construction, min^m phase solution is obtained.

Group Delay

The phase response can be either strictly linear or nonlinear.

Suppose the frequency response is "zero phase", i.e., purely real-valued, then we need not bother about phase response.

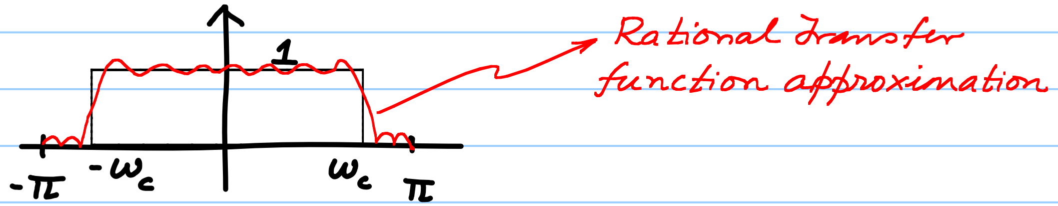
Consider the ideal LPF.



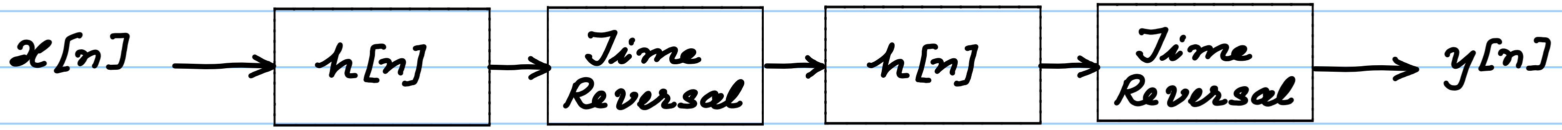
$$\longleftrightarrow \quad \frac{\sin \omega_c n}{\pi n}$$

The above filter is not realizable.

Suppose we approximate the ideal LPF using a rational transfer function, with frequency response shown below:



To realize a filter with "zero phase," assuming real-valued impulse response, consider the following sequence of operations:



It is easy to verify that $Y(e^{j\omega}) = X(e^{j\omega}) \underbrace{|H(e^{j\omega})|^2}_{\text{zero phase filter}}$

Unfortunately, the above sequence of operations results in a non-causal filter, and hence not realizable.

Instead of zero phase, if we had linear phase, the output of the linear phase filter will be a delayed version of zero phase filter's output. Although delay is a distortion in the strict sense, it is a benign one.

If rational transfer function approximations with linear phase are realizable, then they are what will be implemented in practice.

Let $\text{Cos}\,\omega_1 n + \text{Cos}\,\omega_2 n$ be an input to a filter.

Recall the following result:

$$\text{Cos}\,\omega_0 n \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow |H(e^{j\omega_0})|\,\text{Cos}\left(\omega_0 n + \measuredangle H(e^{j\omega_0})\right)$$

If we want only delay distortion, i.e., output can, at the worst, only be a delayed version of the input, then

$$y[n] = x[n-\alpha] = \cos(\omega_0 \overline{n-\alpha})$$

$$= \cos(\omega_0 n - \omega_0 \alpha)$$

This means, $|H(e^{j\omega_0})| = 1$ and also $\measuredangle H(e^{j\omega_0}) = -\alpha\omega_0$

That is, the phase response must be proportional to frequency, apart from unity gain at that frequency.

When there are two components, for delay distortion,

$$y[n] = \cos(\omega_1 \overline{n-\alpha}) + \cos(\omega_2 \overline{n-\alpha})$$

$$= \cos(\omega_1 n - \omega_1 \alpha) + \cos(\omega_2 n - \omega_2 \alpha)$$

where once again the phase shift has to be proportional to frequency, i.e., linear.

Suppose a filter has gain $|H(e^{j\omega_i})| = 1$ for $i = 1, 2,$ but the phase response is not linear. The output $y[n]$ will be

$$y[n] = \cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2)$$

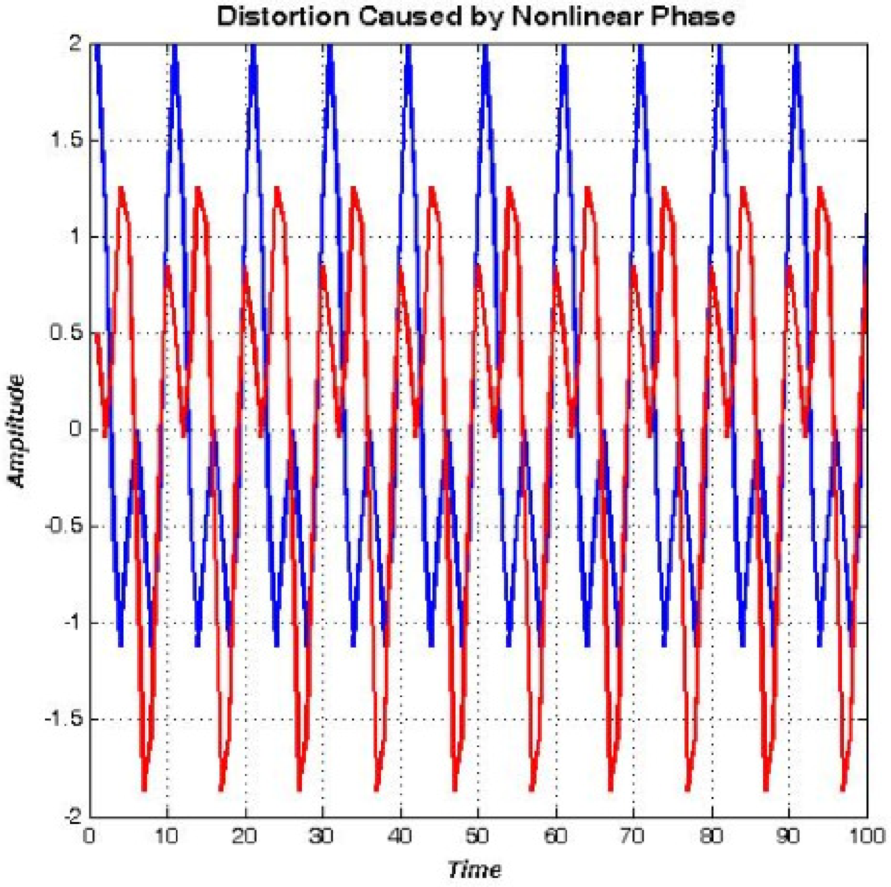
where θ_i is not proportional to ω_i.

Will the waveshape be preserved?

EE5330 Oct. 7, 2013

Note Title 07-10-2013

If the phase is not linear, waveshape will not be preserved. The blue curve is $\cos \omega_1 n + \cos \omega_2 n$. The red curve is $\cos(\omega_1 n + \theta_1) + \cos(\omega_2 n + \theta_2)$ where θ_i is not proportional to ω_i: $\boxed{\text{Waveshape is not preserved.}}$ If $\theta_i \propto \omega_i$, it will cause mere delay.



Linear phase with slope $-L$ will cause a delay of L samples.

The slope, in general, is not constrained to be an integer. What is the meaning of a slope that introduces non-integer delay? Assuming a sampling period T, a fractional delay of $L+\delta$ means that the output is the sampled version of the underlying continuous-time signal delayed by $(L+\delta)T$.

If $$H(e^{j\omega}) = \begin{cases} e^{-j\omega\overbrace{(L+\delta)}^{\text{non-integer}}} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

and the input $X(e^{j\omega})$ has components only in the passband, the output is given by

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega(n-L-\delta)}\, d\omega$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, \text{Sinc}(\overline{n-L-\delta} - m)$$

Group Delay

$$\tau_g(\omega) \triangleq -\frac{d}{d\omega}\phi(\omega)$$ where $\phi(\omega)$ is the continuous phase function

Phase Delay

$$\tau_p(\omega) \triangleq -\frac{\phi(\omega)}{\omega}$$

If $\phi(\omega) = -\alpha\omega$, then $\tau_g(\omega) = \tau_p(\omega) = \alpha$ [constant!]

$\tau_g(\omega)$ and $\tau_p(\omega)$ have the following interprotation:

If a narrowband signal is passed through a narrowband filter, the envelope of the output gets delayed by $\tau_g(\omega_0)$ and the carrier suffers a phase lag of $\tau_p(\omega_0)$, where ω_0 is the centre frequency.

Since $\phi(\omega) = \tan^{-1}\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right]$, it is easy to see that

$$\tau_g(\omega) = \frac{H_I(e^{j\omega})H_R'(e^{j\omega}) - H_R(e^{j\omega})H_I'(e^{j\omega})}{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})} = -\text{Im}\left\{\frac{H'(e^{j\omega})}{H(e^{j\omega})}\right\}$$

For a single complex zero,

$$\phi(\omega) = \tan^{-1} \frac{r \sin(\omega-\theta)}{1- r \cos(\omega-\theta)}$$

$$\Rightarrow \quad \tau_g(\omega) = \frac{r - \cos(\omega-\theta)}{r + \frac{1}{r} - 2\cos(\omega-\theta)}$$

If $r < 1$ and we replace $re^{j\theta}$ by $\frac{1}{r} e^{j\theta}$, then the above expression reveals that the group delay increases.

That is, reflecting an inside-unit-circle zero about the unit circle s.t. it now lies outside increases the group delay.

Units of $\tau_g(\omega)$ are samples

For real-valued $h[n]$, $\phi(\omega) = -\phi(-\omega) \Rightarrow \tau_g(\omega)$ is an even function.
[makes sense since delay at ω and $-\omega$ must be the same]

$\tau_g(\omega) > 0$ in the passband of causal, stable filters

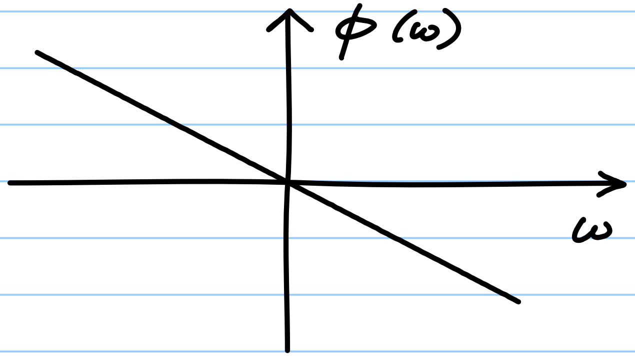
$\tau_g(\omega)$ can assume any real-value, not necessarily an integer.

In general, $\tau_g(\omega)$ is a nolinear function.

A rapid change in phase, typically caused by poles or zeros close to the unit circle, will cause a spike in $\tau_g(\omega)$.

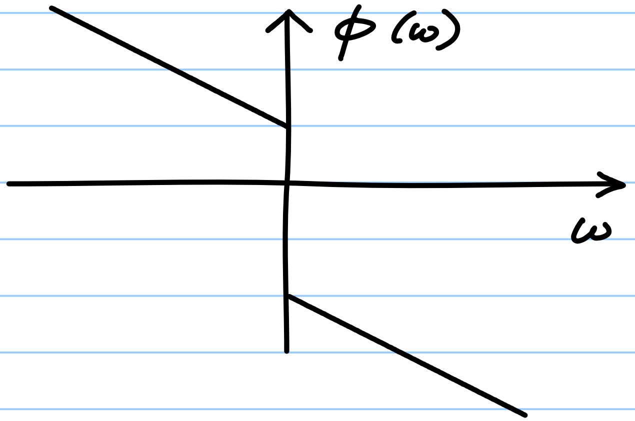
Since linear phase is essential for preserving waveshape, we will examine its consequences in more detail.

Linear Phase



$$\phi(\omega) = -\omega\tau_g$$

Generalized Linear Phase



$$\phi(\omega) = \beta - \omega\tau_g$$

Linear phase is a special case of generalized linear phase with $\beta = 0$.

For both case $\tau_g(\omega) = \tau_g$, a constant. But,

$$\tau_p(\omega) = \begin{cases} \tau_g & \text{for linear phase} \\ \tau_g - \frac{\beta}{\omega} & \text{for generalized linear phase} \end{cases}$$

What constraints, if any, are imposed on filters with linear phase ?

EE5330 Oct. 8, 2013

Note Title 08-10-2013

Periodicity of $H(\omega)$, i.e., $H(\omega) = H(\omega+2\pi)$ and real-valuedness of the impulse response, i.e, $h[n] \in \mathbb{R}$, coupled with linear phase, impose some constraints. We will use the $H(\omega)$ notation rather than the usual $H(e^{j\omega})$.

Let $H(\omega) = A(\omega)\, e^{j(\beta - \omega\tau_g)}$

$$H(\omega + 2\pi) = H(\omega)$$

i.e., $A(\omega)\, e^{j(\beta-\omega\tau_g)} = A(\omega+2\pi)\, e^{j(\beta - \omega\tau_g - 2\pi\tau_g)}$

$$\Rightarrow \quad A(\omega) = A(\omega+2\pi)\, e^{-j2\pi\tau_g}$$

Since $A(\omega) \in \mathbb{R}$, $2\tau_g \in \mathbb{Z}$

(1) $\tau_g = \underbrace{M}_{\text{integer}} \Rightarrow A(\omega) = A(\omega + 2\pi)$ periodic with period 2π

(2) $\tau_g = \underbrace{M + \frac{1}{2}}_{\text{integer} + \frac{1}{2}} \Rightarrow A(\omega) = -A(\omega + 2\pi)$ periodic with period 4π

Also, since $h[n] \in \mathbb{R}$, $H(\omega) = H^*(-\omega)$. Hence,

$$A(\omega) e^{j(\beta - \omega\tau_g)} = A(-\omega) e^{-j(\beta + \omega\tau_g)}$$

$$\Rightarrow \quad \frac{A(\omega)}{A(-\omega)} = e^{-j2\beta}$$

Since $\frac{A(\omega)}{A(-\omega)} \in \mathbb{R}$, $2\beta = 0$ or $\frac{\pi}{2}$

(or π) (or $\frac{3\pi}{2}$)

(1) If $\beta = 0$, $A(\omega) = A(-\omega)$ Even symmetry

(2) If $\beta = \frac{\pi}{2}$ $A(\omega) = -A(-\omega)$ Odd symmetry

Thus, overall, we have FOUR possibilities:

$\tau_g = M$	$\beta = 0$	$A(\omega) = A(-\omega)$	Integer group delay
$\tau_g = M + \frac{1}{2}$	$\beta = 0$	$A(\omega) = A(-\omega)$	Integer $+\frac{1}{2}$ group delay
$\tau_g = M$	$\beta = \frac{\pi}{2}$	$A(\omega) = -A(-\omega)$	Integer group delay
$\tau_g = M + \frac{1}{2}$	$\beta = \frac{\pi}{2}$	$A(\omega) = -A(-\omega)$	Integer $+\frac{1}{2}$ group delay

Suppose we further assume that linear phase filter is CAUSAL. $h[n] = 0$ for $n < 0$. First consider the case $\beta = 0$.

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)e^{-j\omega\tau_g}\, e^{j\omega n}\,d\omega$$

$$h^*[2\tau_g - n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\, e^{j\omega\tau_g}\, e^{-j\omega(2\tau_g - n)}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\,e^{-j\omega\tau_g}\, e^{j\omega n}\,d\omega = h[n]$$

That is, for $\beta = 0$, $h[n] = h^*[2\tau_g - n]$ $\boxed{\text{for ANY linear phase filter.}}$

Hence, if we further assume $h[n] = 0$ for $n < 0$, $h^*[2\tau_g - n] = 0$ for $n > 2\tau_g$ $\Rightarrow$ $\boxed{\text{the filter is FIR}}$

For $\beta = \frac{\pi}{2}$, show that the condition to be satisfied is
$h[n] = -h^*[2\tau_g - n]$

$$\boxed{\begin{array}{l} h[n] = h^*[2\tau_g - n] \Rightarrow \text{symmetry around } n = \tau_g \\ h[n] = -h^*[2\tau_g - n] \Rightarrow \text{anti-symmetry around } n = \tau_g \end{array}}$$

Let the FIR filter be defined over the interval $n = 0, 1, \ldots, N-1$. Hence $h[n] = 0$ for $n < 0$ and $n > N-1$. Hence $2\tau_g = N-1$, i.e.,

$$\boxed{\tau_g = \frac{N-1}{2}}.$$

Therefore, if the length (N) of the filter is odd, τ_g is an integer; if the length is even, τ_g equals integer $+ \frac{1}{2}$ samples.

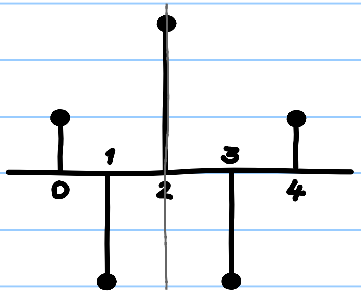
Therefore, the delay introduced by a linear phase FIR filter is either integer or integer $+ \frac{1}{2}$ samples.

Length	Symmetry	Group Delay	Name
Odd	Even $\beta = 0$	$\frac{N-1}{2}$ int	Type I
Even	Even $\beta = 0$	$\frac{N-1}{2}$ int $+ \frac{1}{2}$	Type II
Odd	Odd $\beta = \frac{\pi}{2}$	$\frac{N-1}{2}$ int	Type III
Even	Odd $\beta = \frac{\pi}{2}$	$\frac{N-1}{2}$ int $+ \frac{1}{2}$	Type IV

EE5330 Oct. 10, 2013

Since $2\tau_g = N-1$, for an FIR filter with real-valued coefficients,

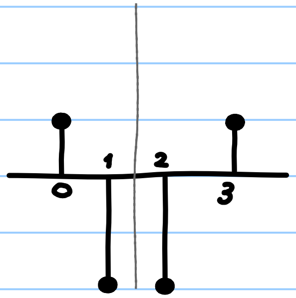
$h[N-1-n] = \pm h[n]$



Type I

$N = 5$

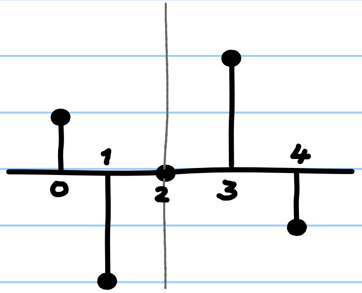
$\tau_g = \frac{5-1}{2} = 2$



Type II

$N = 4$

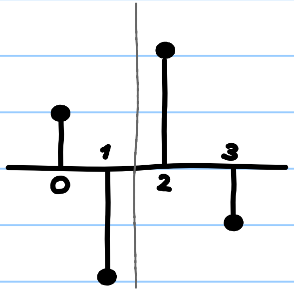
$\tau_g = \frac{4-1}{2} = 1.5$



Type III

$N = 5$

$\tau_g = \frac{5-1}{2} = 2$



Type IV

$N = 4$

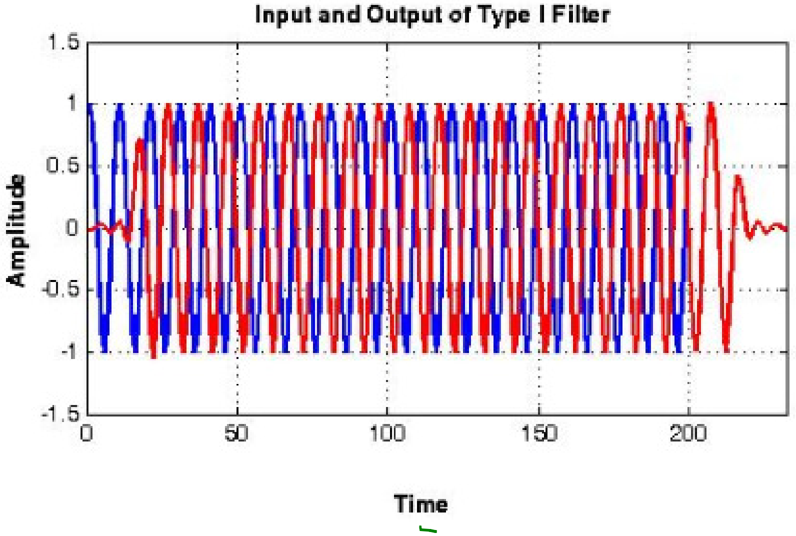
$\tau_g = \frac{4-1}{2} = 1.5$

If a filter produces integer delay, the output can be shifted to the left by that many samples and the original and filtered signals can be time-aligned.

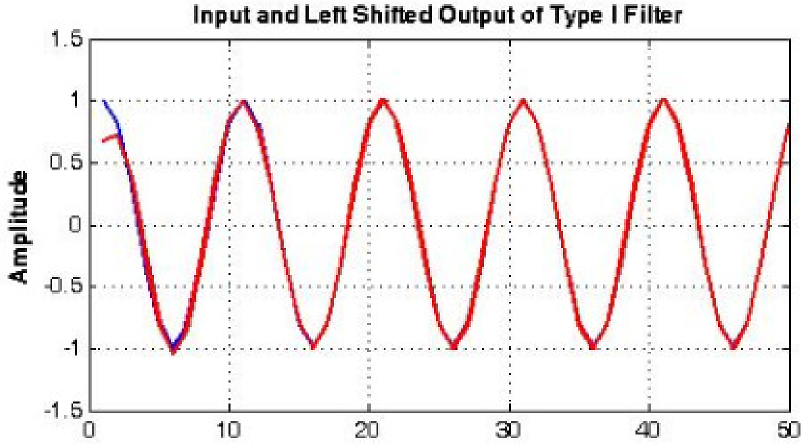
If the filter introduces half-sample delays, the time-alignment is not possible since the shift can only be an integer.

Types I and III introduce integer delays

Types II and IV introduce half-sample delays.

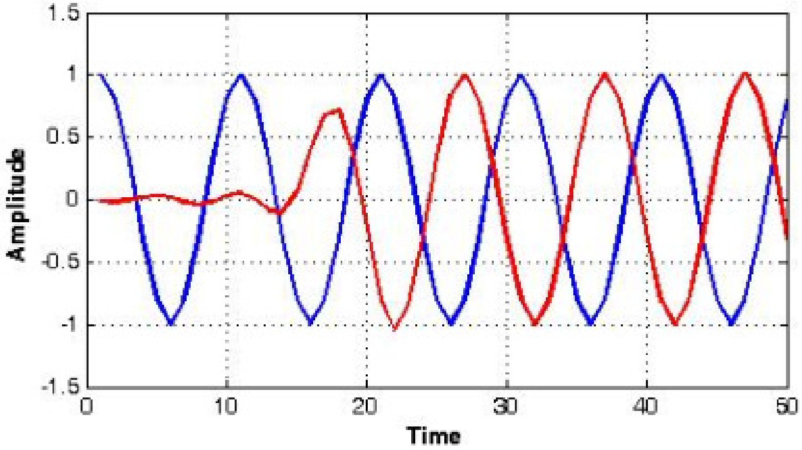


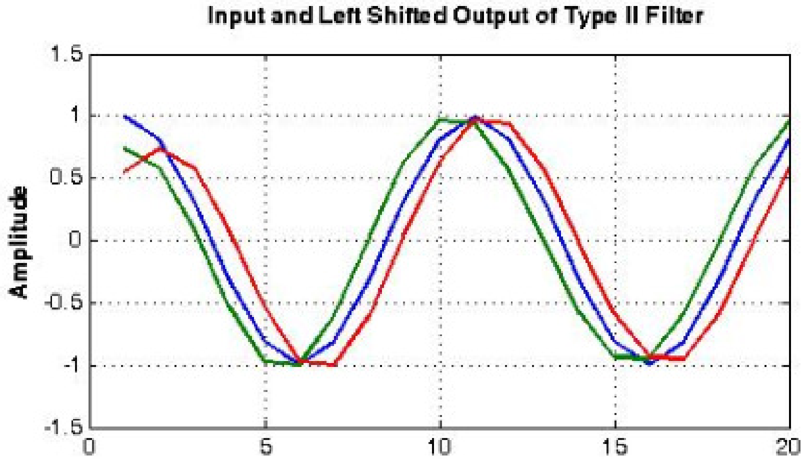
Input and Output of Type I Filter



Input and Left Shifted Output of Type I Filter

Integer shift equal to 0.5 x (N-1) produces exact alignment with the input





Input and Left Shifted Output of Type II Filter

Neither of the integer shifts closest to 0.5 x (N-1) produces exact alignment with the input

For Types I and III, the centre of symmetry falls on a sample

For Types II and IV the centre of symmetry falls midway between samples.

Symmetry is both necessary and sufficient for an FIR filter to be linear phase.

Symmetry is sufficient but not necessary for an IIR filter to be linear phase

$$h[n] = \frac{\sin w_c(n-\alpha)}{\pi(n-\alpha)}$$ is linear phase for any α.

However $h[n]$ is symmetric around $n=\alpha$ only if α is integer or integer $+\frac{1}{2}$.

Frequency Response of Linear Phase FIR Filters

$$H(\omega) = \sum_{n=0}^{\boxed{N-1} \rightarrow M} h[n]e^{-j\omega n}.$$ It is usual to let $M=N-1$.

For Type I, $h[0]=h[M]$, $h[1]=h[M-1]$, and so on. Hence,

$$H(\omega) = h[0]+h[1]e^{-j\omega}+\dots+h[M-1]e^{-j(M-1)\omega}+h[M]e^{-jM\omega}$$

$$= h[0]+h[1]e^{-j\omega}+\dots+h[1]\,e^{-j(M-1)\omega}+h[0]\,e^{-jM\omega}$$

$$= e^{-j\omega M/2} \underbrace{\left\{ h\left[\frac{M}{2}\right] + \sum_{n=0}^{\frac{M}{2}-1} 2h[n] \cos\left(\overline{\frac{M}{2} - n}\, \omega \right) \right\}}_{A(\omega)}$$

Similarly,

$$H(\omega) = e^{-j\omega M/2} \left\{ \sum_{n=0}^{\frac{M-1}{2}} 2h[n] \cos\left(\overline{\frac{M}{2} - n}\, \omega \right) \right\}$$

$$H(\omega) = \boxed{j}\, e^{-j\omega M/2} \left\{ \sum_{n=0}^{\frac{M}{2}-1} 2h[n] \sin\left(\overline{\frac{M}{2} - n}\, \omega \right) \right\}$$

β

$e^{j\boxed{\frac{\pi}{2}}} \leftarrow$

$$H(\omega) = \boxed{j}\, e^{-j\omega M/2} \left\{ \sum_{n=0}^{\frac{M-1}{2}} 2h[n] \sin\left(\overline{\frac{M}{2} - n}\, \omega \right) \right\}$$

EE5330 Oct. 14, 2013

Zero Locations of Linear Phase FIR Filters:

Recall that linear phase imposes the following condition:

$h[n] = \pm h^*[M-n]$ where $M = N-1$

Hence

$$H(z) = \pm z^{-M} H^*(1/z^*)$$

Suppose z_0 is a zero of $H(z)$. That is, $H(z_0) = 0$.

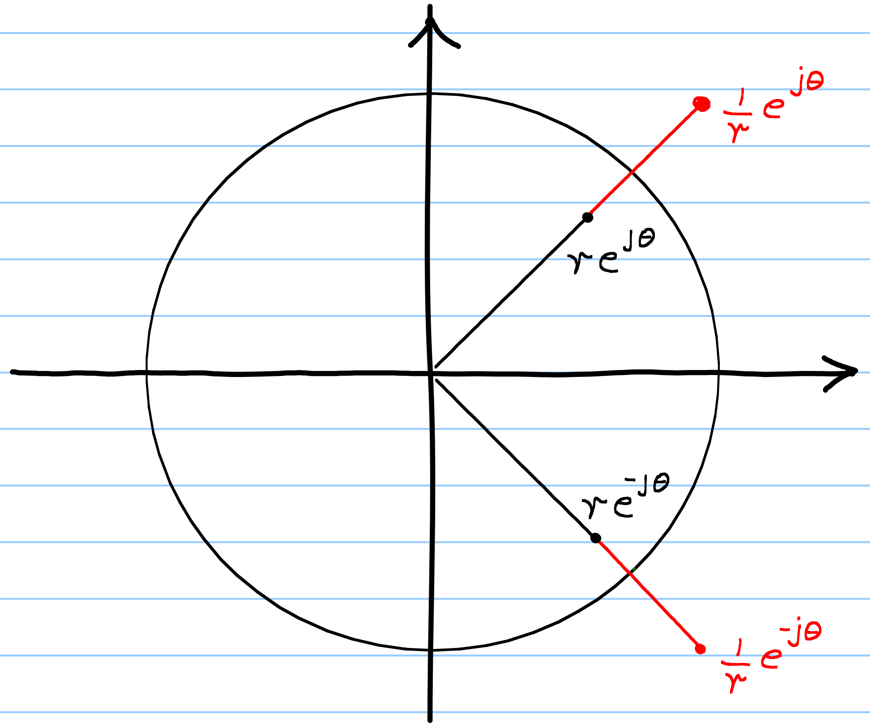
This means that

$$H(z_0) = 0 = z_0^{-M} H^*(1/z_0^*) \Rightarrow 1/z_0^* \text{ is also a zero}$$

Thus, if $re^{j\theta}$ is a zero, then $\frac{1}{r}e^{j\theta}$ is also a zero.

If $h[n] \in \mathbb{R}$, then $re^{-j\theta}$ will also be a zero $\Rightarrow$ $\frac{1}{r}e^{-j\theta}$ will be a zero too. Thus, a complex zero that is not on the unit circle must occur in $\boxed{\text{sets of 4}}$ for a linear phase FIR filter with real-valued impulse response.

If $r = 1$, the $\boxed{\text{same zero}}$ satisfies both $H(z_0) = 0$ and the $H(1/z_0^*) = 0$.



$h[n] \in \mathbb{R}$ and linear phase mean that, if $re^{j\theta}$ is a zero, then the set of related zeros is $\{ re^{\pm j\theta}, \frac{1}{r}e^{\pm j\theta} \}$

Linear phase filters also have constrained zeros.

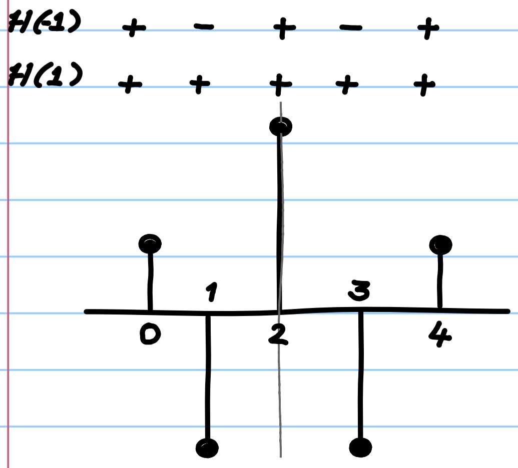
For an FIR filter $h[n]$,

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n}$$

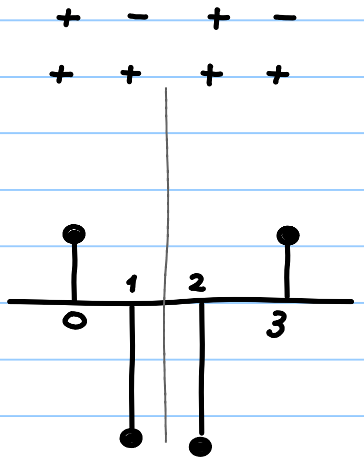
We will examine $H(1)$ and $H(-1)$.

$$H(1) = \sum_{n=0}^{N-1} h[n]$$

$$H(-1) = \sum_{n=0}^{N-1} h[n](-1)^{n}$$

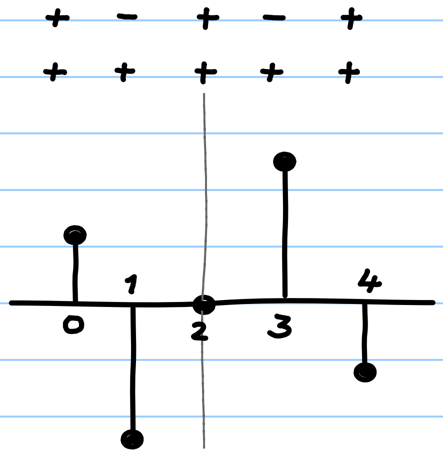


Type I



Type II

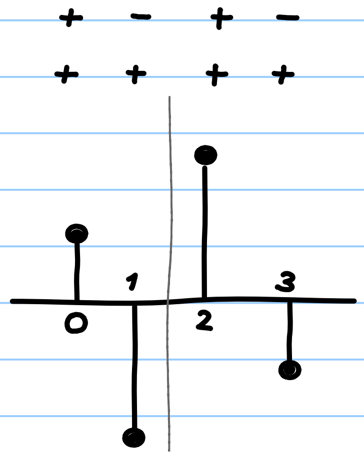
$H(-1) = 0$
always



Type III

$H(1) = 0$
always

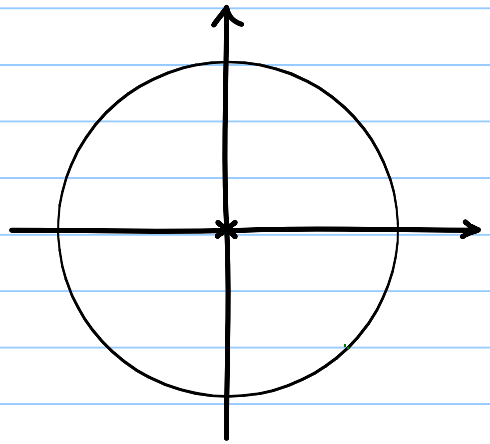
$H(-1) = 0$
always



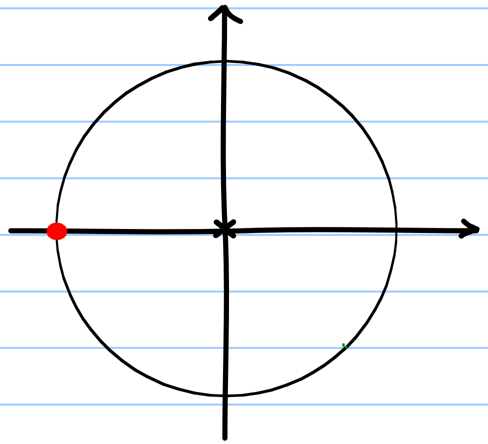
Type IV

$H(1) = 0$
always

Type I

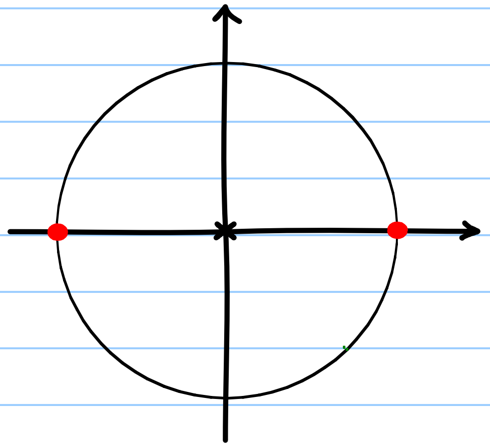


Type II



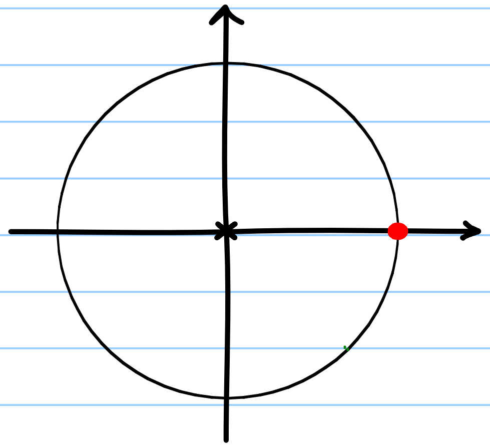
Cannot be used for building HPF

Type III



Cannot be used for building LPF, HPF

Type IV



Cannot be used for building LPF

EE5330 Oct. 15, 2013

Causal, stable, linear phase filters with rational transfer functions have to be necessarily FIR.

We will study more about all-pass and min^m phase filters.

A K^{th} order all-pass filter can be written as a cascade of K first order all-pass sections.

$$H_k(z) = \frac{-a_k^* + z^{-1}}{1 - a_k z^{-1}} = \frac{z^{-1} - r_k e^{-j\theta_k}}{1 - r_k e^{j\theta_k} z^{-1}}$$

$$H_{ap}(z) = \prod_{k=1}^{K} H_k(z)$$

$$\measuredangle H_k(e^{j\omega}) = \phi_k(\omega) = \arg\{e^{-j\omega} - r_k e^{-j\theta_k}\} - \arg\{1 - r_k e^{j\theta_k} e^{-j\omega}\}$$

$$= \arg\{e^{-j\omega}\} + \arg\{1 - r_k e^{-j\theta_k} e^{j\omega}\} - \arg\{1 - r_k e^{j\theta_k} e^{-j\omega}\}$$

$$= -\omega - 2\tan^{-1} \frac{r_k \sin(\omega - \theta_k)}{1 - r_k \cos(\omega - \theta_k)}$$

The overall phase response is,

$$\phi(\omega) = -K\omega - 2\sum_{k=1}^{K} \tan^{-1} \frac{r_k \sin(\omega - \theta_k)}{1 - r_k \cos(\omega - \theta_k)}$$

The associated group delay is

$$\tau_g(\omega) = -\frac{d}{d\omega}\phi(\omega)$$

$$= K + 2\sum_{k=1}^{K} \frac{r_k \cos(\omega-\theta_k) - r_k^2}{1 - 2r_k \cos(\omega-\theta_k) + r_k^2}$$

$$= \sum_{k=1}^{K} \frac{1 - r_k^2}{1 - 2r_k \cos(\omega-\theta_k) + r_k^2} = \sum_{k=1}^{K} \frac{1 - r_k^2}{\left|1 - r_k e^{j\theta_k} e^{-j\omega}\right|^2}$$

Since $r_k < 1 \ \forall k$, $\tau_g(\omega) > 0$ for an all-pass filter

Also, since $\tau_g(\omega) = -\phi'(\omega)$, $\phi(\omega)$ is a monotonic decreasing function.

One can also easily prove the following:

$$|H_k(z)| = \begin{cases} > 1 & |z| < 1 \\ = 1 & |z| = 1 \\ < 1 & |z| > 1 \end{cases} \Rightarrow \text{this property holds for } \underbrace{H_{ap}(z)}_{K^{th} \text{ order all-pass}} \text{ also}$$

We also saw that $H(z)$ is called as a minimum phase filter if all its poles and zeros are inside the unit circle.

To see the connection between a general $H(z)$ and its associated minimum phase and all-pass decomposition, let $H(z)$ be such that it has only one zero outside the unit circle.

$$H(z) = H_1(z)\left(z^{-1} - c_k^*\right)$$

That is, the zero is at $\frac{1}{c_k^*}$, where $|c_k| < 1$

Rewrite $H(z)$ as follows:

$$H(z) = H_1(z)(1-c_k z^{-1}) \cdot \frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}$$

Since $|c_k| < 1$, $H_1(z)(1-c_k z^{-1})$ is minimum phase and

$\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}$ is all-pass.

This procedure can be repeated for every outside-unit-circle zero, and hence any $H(z)$ can be written as $H_{min}(z) \cdot H_{ap}(z)$

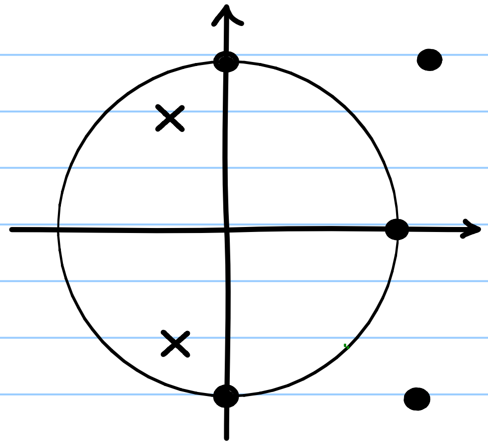
If $H(z)$ has zeros on the unit circle, then those zeros cannot be part of $H_{min}(z)$. Hence, the most general decomposition of $H(z)$ is as follows:

$$H(z) = H_{min}(z) \cdot H_{uc}(z) \cdot H_{ap}(z)$$

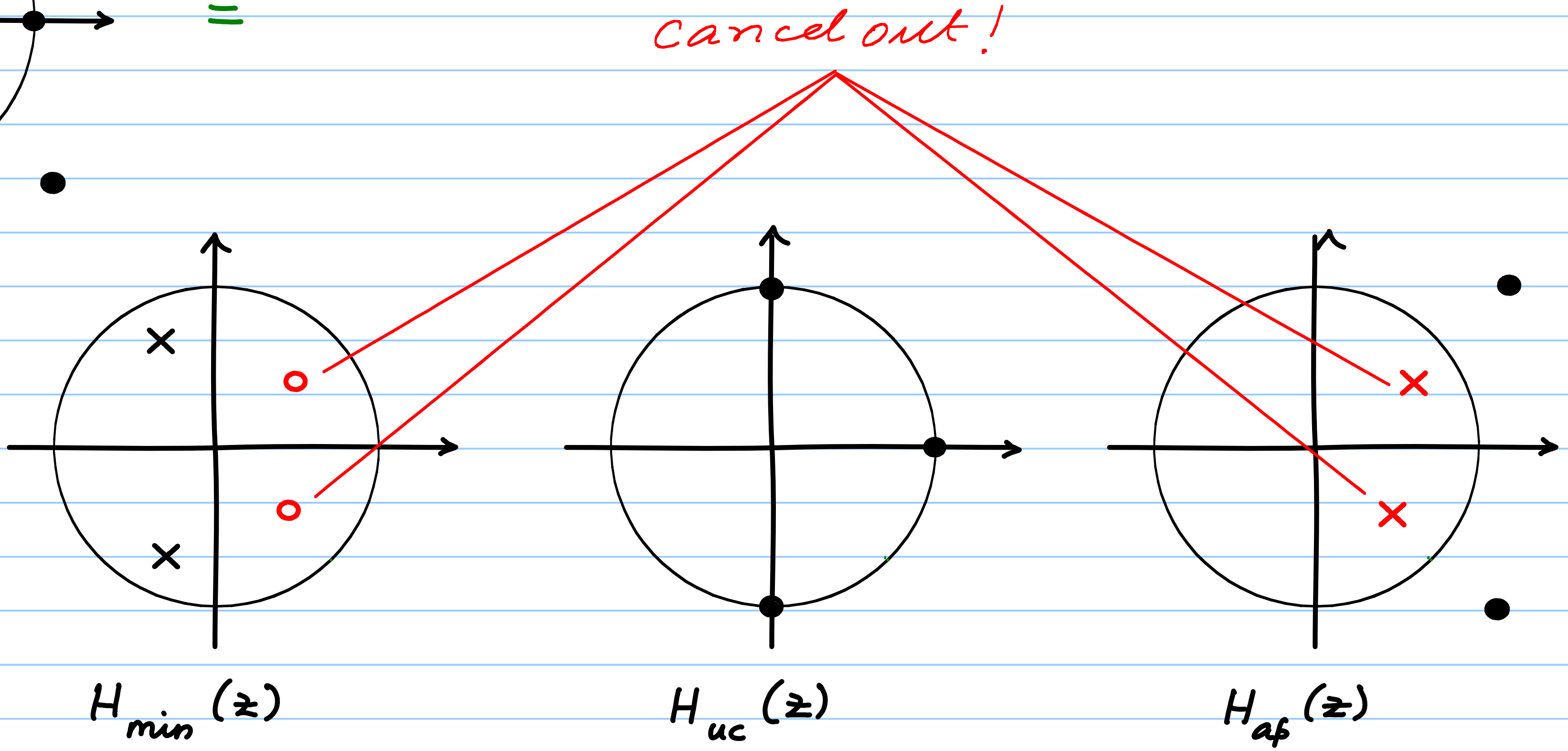
where $H_{uc}(z)$ contains all the unit circle zeros of $H(z)$.

If a system is minimum phase, causal, and stable, its inverse system is also causal, stable, and minimum phase.

$$\boxed{H(z) = H_{min}(z)\, H_{uc}(z)\, H_{ap}(z)}$$



=



EE5330 Oct. 21, 2013

An all-pass filter preserves signal energy.

$$x[n] \longrightarrow \boxed{H_{ap}(z)} \longrightarrow y[n]$$

$$\| \underline{x} \|_2^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\| \underline{y} \|_2^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega$$ [Parseval's Theorem]

$$Y(e^{j\omega}) = X(e^{j\omega})\, H_{ap}(e^{j\omega})$$

$$|Y(e^{j\omega})|^2 = |X(e^{j\omega})|^2 \cdot \cancel{|H_{ap}(e^{j\omega})|^2} = |X(e^{j\omega})|^2$$

Hence, since $|X(e^{j\omega})|^2 = |Y(e^{j\omega})|^2$,

$$\|\underline{x}\|_2^2 = \|\underline{y}\|_2^2 \quad \text{i.e.,} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2$$

That is, the $\boxed{\text{all-pass filter preserves energy.}}$

We will now prove the following stronger result:

$$\sum_{n=-\infty}^{n_0} |x[n]|^2 \geq \sum_{n=-\infty}^{n_0} |y[n]|^2$$

i.e., the running sum of the output energy of an all-pass filter is always less than or equal to the corresponding i/p energy sum.

EE5330 Oct.22, 2013

Consider the following causal and stable all-pass filter:

We showed earlier that $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |y[n]|^2$

Since the filter is causal, and the i/p is applied at $n=0$, the lower limit can be replaced by $n=0$.

Now consider the following i/p: $x_1[n] = \begin{cases} x[n] & n \le n_0 \\ 0 & n > n_0 \end{cases}$

Let $y_1[n]$ be the corresponding output. Then,

$$\sum_{n=0}^{\infty} |x_1[n]|^2 = \sum_{n=0}^{\infty} |y_1[n]|^2$$

$$\sum_{n=0}^{n_0} |x[n]|^2 = \sum_{n=0}^{n_0} |y_1[n]|^2 + \sum_{n=n_0+1}^{\infty} |y_1[n]|^2$$

$$= \sum_{n=0}^{n_0} |y[n]|^2 + \sum_{n=n_0+1}^{\infty} |y_1[n]|^2$$

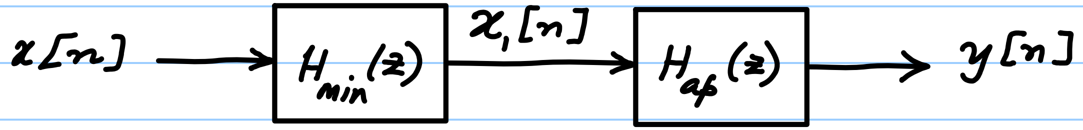
[Since $y_1[n] = y[n]$ for $n \leq n_0$]

$$\geq \sum_{n=0}^{n_0} |y[n]|^2$$

Thus, for an all-pass filter, $\boxed{\sum_{n=0}^{n_0} |x[n]|^2 \geqslant \sum_{n=0}^{n_0} |y[n]|^2}$

The above result can be used to show that min^m phase filters have the least energy delay.

Recall that any $H(z)$ can be decomposed as follows:



$y[n]$ is the o/p of an arbitrary, causal, stable filter

$x_1[n]$ is the o/p of the minimum phase counterpart of $H(z)$.

Using the previous result,

$$\sum_{n=0}^{n_0} |x_1[n]|^2 \geqslant \sum_{n=0}^{n_0} |y[n]|^2$$

That is, the minimum-phase filter has the least energy lag.

Hence the term "minimum lag" is more accurate than the well-entrenched "minimum phase" terminology.

"Causal" DTFT and its implications

Recall that $x[n] = 0$ for $n < 0$ imposed restrictions on the corresponding transform's real and imaginary parts.

Suppose now that $X(e^{j\omega}) = 0$ for $\omega < 0$, i.e., $X(e^{j\omega})$ is "causal".
Since $X(e^{j\omega})$ is periodic, "causal" here means $X(e^{j\omega}) = 0$ for $-\pi < \omega < 0$. Similar to expressing $x[n]$ as $x_e[n] + x_o[n]$, consider

$$X_e(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right]$$

$$X_o(e^{j\omega}) = \frac{1}{2j}\left[X(e^{j\omega}) - X^*(e^{-j\omega})\right]$$

We can recover $X(e^{j\omega})$ over $0<\omega<\pi$ from either $X_e(e^{j\omega})$ or $X_o(e^{j\omega})$:

$$X(e^{j\omega}) = \begin{cases} 2X_e(e^{j\omega}) & 0<\omega<\pi \\ 0 & -\pi<\omega<0 \end{cases}$$

$$X(e^{j\omega}) = \begin{cases} 2j\,X_o(e^{j\omega}) & 0<\omega<\pi \\ 0 & -\pi<\omega<0 \end{cases}$$

One can also relate $X_e(e^{j\omega})$ and $X_o(e^{j\omega})$.

It is easy to see that

$$X_o(e^{j\omega}) = \begin{cases} -jX_e(e^{j\omega}) & 0 < \omega < \pi \\ jX_e(e^{j\omega}) & -\pi < \omega < 0 \end{cases}$$

That is,

$$X_o(e^{j\omega}) = X_e(e^{j\omega})\, H(e^{j\omega})$$

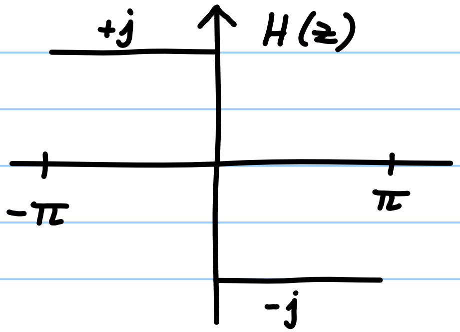
where

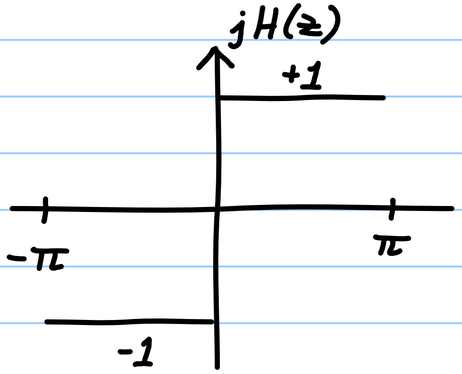
$$H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

Note that $x[n] = x_R[n] + j\,x_I[n]$

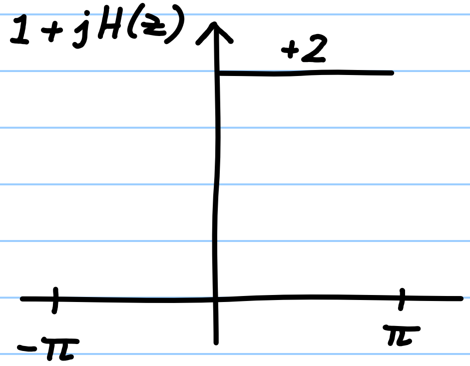
$$x_R[n] \longleftrightarrow X_e(e^{j\omega})$$

$$x_I[n] \longleftrightarrow X_o(e^{j\omega})$$



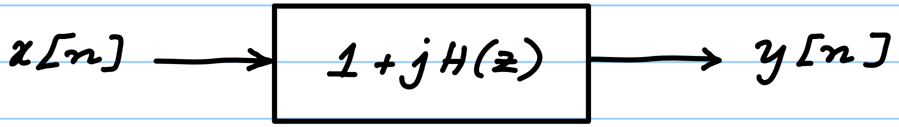


Complex Half Band Filter



Let $G(z) = 1 + jH(z)$, whence it follows $G(e^{j\omega}) = \begin{cases} 2 & 0 < \omega < \pi \\ 0 & -\pi < \omega < 0 \end{cases}$

Hence if an arbitrary $x[n]$ is filtered using $G(e^{j\omega})$, the output signal's DTFT becomes "causal" (or "one-sided").



$$g[n] = \delta[n] + jh[n]$$

Hence,

$$y[n] = x[n] * g[n]$$

$$y[n] = (\delta[n] + jh[n]) * x[n]$$

$$= x[n] + j\,x[n] * h[n]$$

$$= x[n] + j\,\hat{x}[n] = x_R[n] + j\,x_I[n] \Rightarrow \boxed{x_R[n] \;\&\; x_I[n] \text{ are not independent}}$$

where $\hat{x}[n] = x[n] * h[n]$

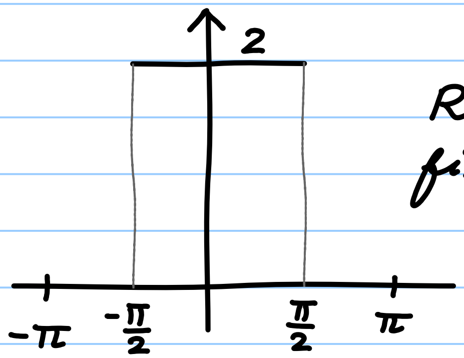
Since $H(e^{j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$, one can easily verify that

$$h[n] = \begin{cases} \dfrac{\sin^2(n\pi/2)}{n\pi/2} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

$\boxed{h[n] \leftrightarrow H(e^{j\omega}) \text{ is called as the IDEAL HILBERT TRANSFORMER}}$

Exercise

Explore the relationship between the real-valued halfband filter, complex halfband filter, and the hilbert transformer. The response of a real halfband filter is given below.



Real halfband filter

Note Title 23-10-2013

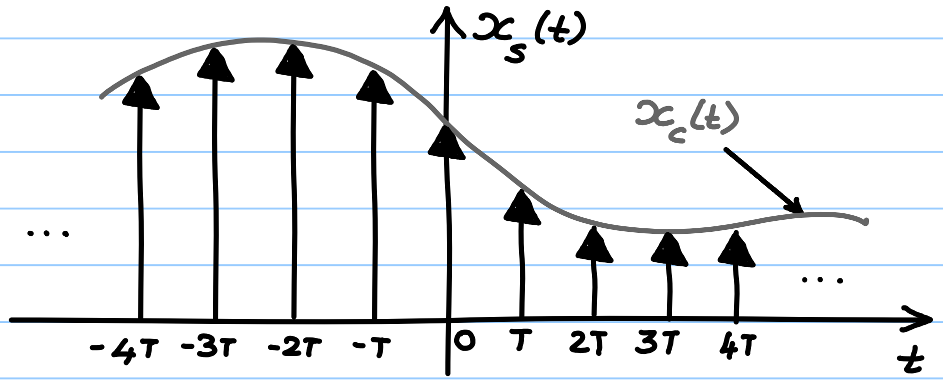
EE5330 Oct. 23, 2013

Sampling:

The process of sampling provides the bridge between the CT and DT domains. To connect the spectrum of a CT signal with that of the DT sequence's spectrum, we use the theoretical framework of impulse-train sampling.

F, Ω : CTFT frequency $\Omega = 2\pi F$

f, ω : DTFT frequency $\omega = 2\pi f$



$$x_c(t) \longleftrightarrow X_c(F)$$

$$x_s(t) = x_c(t) \cdot p(t)$$

where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$$\Rightarrow \quad x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

$$X_s(F) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j2\pi F t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j2\pi\left(\frac{F}{F_s}\right)n}$$ since $F_s = \frac{1}{T}$

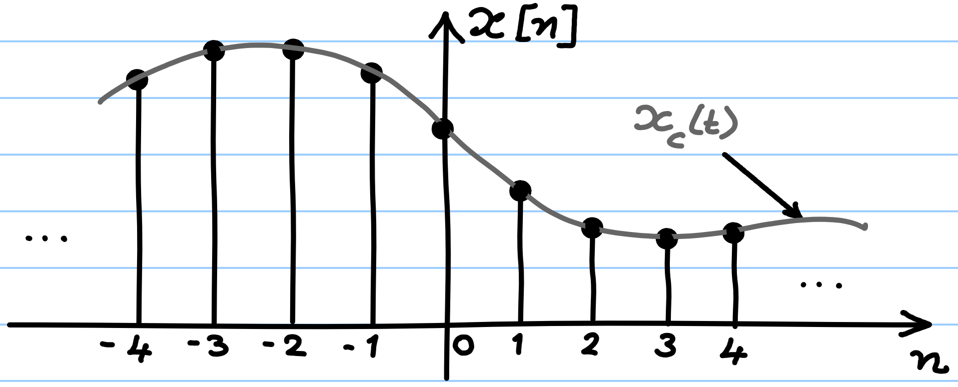
Alternatively, $p(t) \longleftrightarrow P(F) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(F - nF_s)$

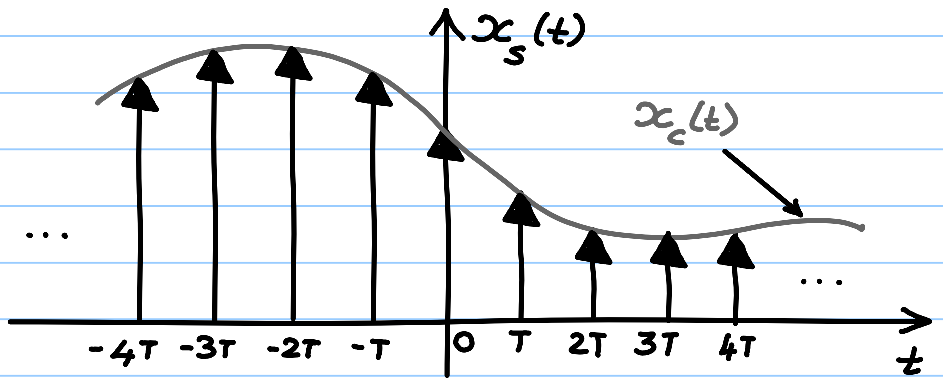
Hence $x_c(t) \cdot p(t) \longleftrightarrow X_c(F) * P(F)$

Thus
$$\boxed{X_s(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(F - kF_s) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j 2\pi \left(\frac{F}{F_s}\right) n}}$$

How does the spectrum of the impulse-train sampled signal relate to the spectrum of a sequence whose values are $x[n] = x_c(nT)$?

That is, are $X(e^{j\omega})$ and $X_s(F)$ related ?





To relate the spectra of the above signals, define $x[n] \triangleq x_c(nT)$. Hence,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

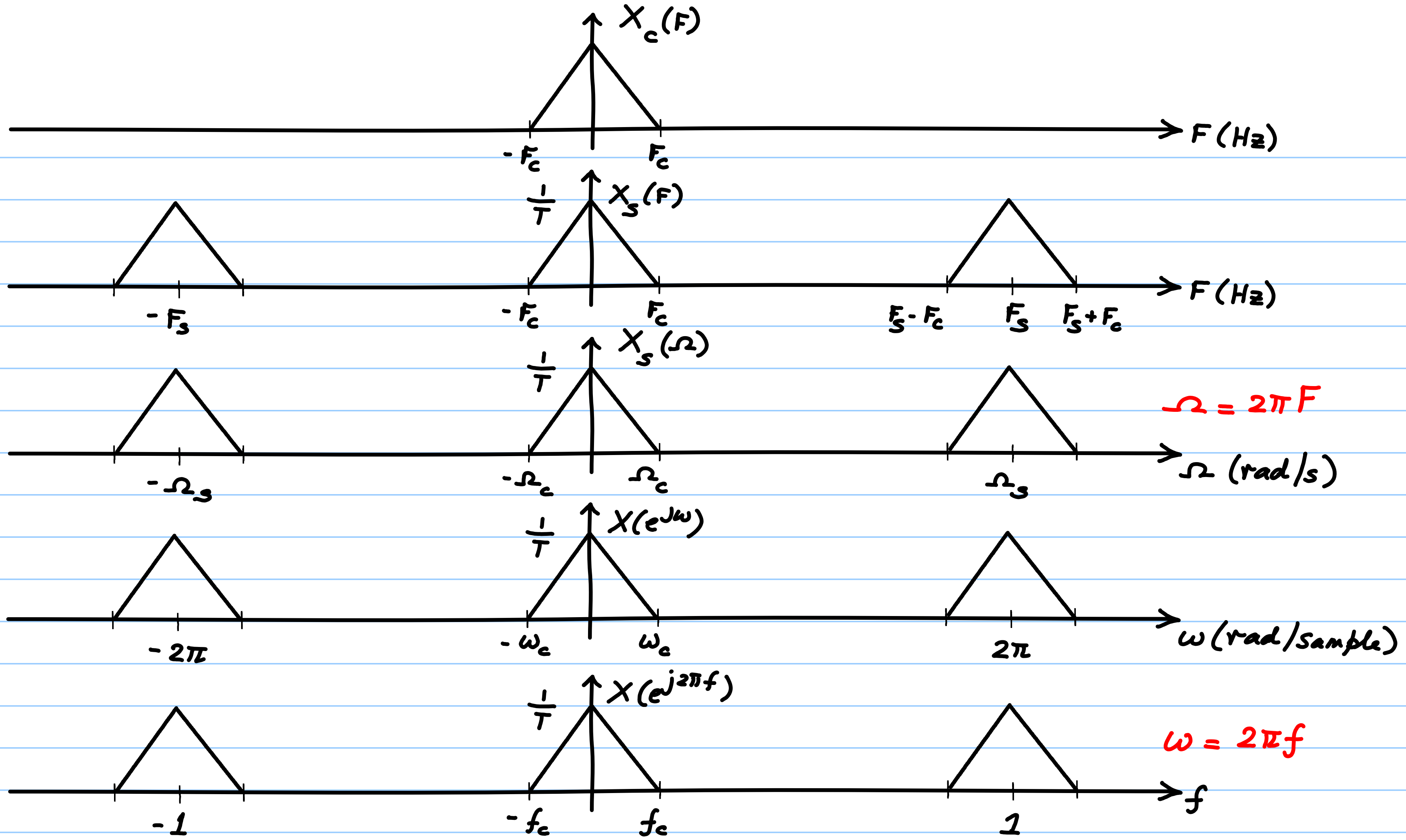
Recall $X_s(F) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j 2\pi \left(\frac{F}{F_s}\right) n}$

Clearly, then,

$$X(e^{j\omega}) = X_s(F)\Big|_{F \to \frac{\omega}{2\pi T}}$$

Since $f = \frac{\omega}{2\pi}$ and $F_s = \frac{1}{T}$, the above change of variable converts the F_s-periodic $X_s(F)$ into the 2π-periodic $X(e^{j\omega})$.

If we plot the DTFT as a function $f = \frac{\omega}{2\pi}$, the period becomes 1. These are summarized in the figure below:



$$\Omega = 2\pi F$$

$$\omega = 2\pi f$$

The change of variable $F \to \frac{\omega}{2\pi T}$ means that $X(e^{j\omega})\Big|_{\omega = 2\pi} = X_s(F_s)$.

That is, $X(e^{j\omega})$ is obtained from $X_s(F)$ by scaling it by F_s.

Thus, an analog frequency F_0 Hz gets mapped to $\boxed{\omega_0 = 2\pi \frac{F_0}{F_s}}$

Note that ω_0 is a dimensionless quantity.

The same analog frequency F_0 Hz gets mapped to a different frequency if F_s changes. In particular, if $F_{s_2} > F_{s_1}$, then

$$2\pi \frac{F_0}{F_{s_2}} < 2\pi \frac{F_0}{F_{s_1}}$$

Thus, a bandlimited spectrum with BW F_c gets mapped to a bandlimited spectrum with BW $\frac{F_c}{F_{s_1}} \cdot 2\pi$ [ω notation]. However, the same analog spectrum gets converted to a spectrum with narrower bandwidth $\frac{F_c}{F_{s_2}} \cdot 2\pi$ if sampled at $F_{s_2} > F_{s_1}$.

Thus, excessively high sampling frequencies leads to excessively narrowband spectra, making processing more difficult (in terms of filters needed, processing speed, etc.)

Example

$x_c(t) = \cos 2\pi F_o t$

$x_c(nT) = \cos 2\pi F_o nT$

$= \cos 2\pi \frac{F_o}{F_s} n$

$= \cos 2\pi f_o n$ where $f_o = \frac{F_o}{F_s}$

$= \cos \omega_o n$

Recall the transform pair $\cos 2\pi F_o t \longleftrightarrow \frac{1}{2}\left[\delta(F-F_o) + \delta(F+F_o)\right]$

We need to derive $X(e^{jw})$ starting from the above $X_c(F)$.

$X_s(F) = \frac{1}{2T}\left[\delta(F-F_0) + \delta(F+F_0)\right] \qquad -\frac{F_s}{2} \leq F \leq \frac{F_s}{2}$

Now consider $\frac{1}{2}\delta(F-F_0)\Big|_{F \to \frac{\omega}{2\pi T}} = \frac{1}{2T}\delta\left(\frac{\omega}{2\pi T} - F_0\right) = \pi\,\delta\left(\omega - 2\pi\frac{F_0}{F_s}\right)$

$= \pi\,\delta(\omega - \omega_0)$

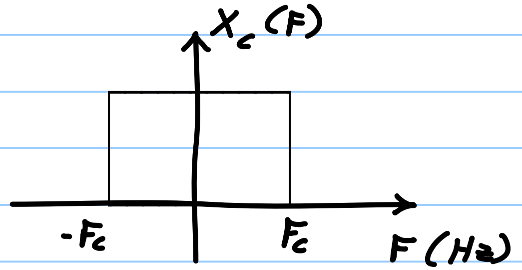
since $\delta(ax+b) = \frac{1}{|a|}\delta\left(x + \frac{b}{a}\right)$ and $\omega_0 = 2\pi f_0 = 2\pi\frac{F_0}{F_s}$

Similarly, $\frac{1}{2T}\delta(F_0 - F_s) \longrightarrow \pi\,\delta(\omega - \omega_0)$.

Thus, $\cos\omega_0 n \longleftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$, as expected.

Example

$$x_c(t) = \frac{\sin 2\pi F_c t}{\pi t}$$

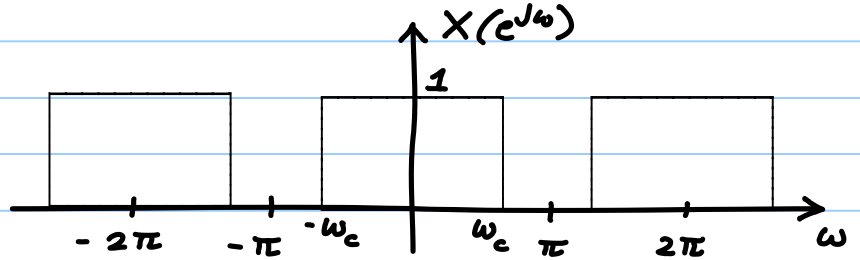


$$X_s(F) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(F - kF_s)$$

$$\frac{1}{T} \frac{\sin 2\pi \frac{F_c}{F_s} n}{\pi n} \longleftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} rect(F - kF_s)$$

Hence

$$\frac{\sin w_c n}{\pi n} \longleftrightarrow$$



Example

$$x_c(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow \text{Sinc}(F) = \frac{\sin \pi F}{\pi F}$$

Since the signal is not bandlimited, sampling will cause aliasing.

We need to derive the pair $x[n] = \begin{cases} 1 & -N \leq n \leq N \\ 0 \end{cases} \longleftrightarrow \frac{\sin (2N+1)\omega/2}{\sin \omega/2}$ starting from the given $x_c(t)$.

$$X_c(F) = \frac{\sin \pi F}{\pi F}$$

$$X_s(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\sin \pi (F - kF_s)}{\pi (F - kF_s)}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \frac{\sin \pi\left(\frac{\omega}{2\pi T} - \frac{k}{T}\right)}{\pi\left(\frac{\omega}{2\pi T} - \frac{k}{T}\right)}$$

$$= \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\omega - 2\pi k}{2T}\right)}{\left(\frac{\omega - 2\pi k}{2}\right)}$$

Thus,

$$\boxed{\frac{\sin (2N+1)\,\omega/2}{\sin \omega/2} = \sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\omega - 2\pi k}{2T}\right)}{\left(\frac{\omega - 2\pi k}{2}\right)}}$$

Thus, the Dirichlet Kernel is the periodic function formed from the analog sinc function.

In the above example we have assumed that none of the sampling points fall on a discontinuity.

Sampling at a discontinuity

Recall that

$$f(t) = \text{P.V.} \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)e^{j\Omega t}dt$$

where P.V. stands for "Principal Value." The LHS is not really $f(t)$, but

$$\frac{f(t^+)+f(t^-)}{2}$$

This equals $f(t)$ only if $f(t)$ is $\boxed{\text{continuous}}$ at t. Otherwise, the inverse transform yields the average of the function values on either side of the discontinuity.

Recall that the sampled signal spectrum is periodic. It can therefore be expressed as a Fourier series. The coefficients of the Fourier series are not arbitrary but closely related to the time function.

Recall the following Fourier series:

$$\sum_{n=-\infty}^{\infty} \delta(\Omega + n\Omega_1) = \frac{1}{\Omega_1} \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1} \quad \text{where} \quad T_1 = \frac{2\pi}{\Omega_1}$$

Hence,

$$F(\Omega) * \sum_{n=-\infty}^{\infty} \delta(\Omega + n\Omega_1) = \frac{1}{\Omega_1} F(\Omega) * \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1}$$

$$\sum_{n=-\infty}^{\infty} F(\Omega + n\Omega_1) = \sum_{k=-\infty}^{\infty} F(\Omega) * e^{-jk\Omega T_1} \cdot \frac{1}{\Omega_1}$$

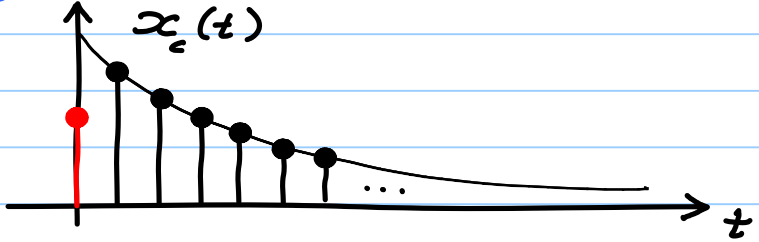
$$= \sum_{k=-\infty}^{\infty} \frac{1}{\Omega_1} \int_{-\infty}^{\infty} e^{-jkT_1(\Omega - y)} F(y) dy$$

$$= \sum_{k=-\infty}^{\infty} e^{-jk\Omega T_1} \frac{1}{\Omega_1} \int_{-\infty}^{\infty} F(y) e^{jkT_1 y} dy$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{\Omega_1} f(kT_1) e^{-jk\Omega T_1}$$

We have to replace the above sample value $f(kT_1)$ with its average value if kT_1 falls on a discontinuity.

Thus if $x_c(t) = e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\Omega}$ and $X(e^{j\omega})$ is is obtained as $\frac{1}{T}\sum_{k=-\infty}^{\infty}\frac{1}{a+j\,2\pi(F-kF_s)}$, the spectrum corresponds to a sequence whose sample value at $n=0$ is $\frac{1}{2}$, and not 1.



EE5330 Oct. 29, 2013

Note Title 29-10-2013

A frequency of F_0 Hz gets mapped to $f = \frac{F_0}{F_s}$ in the DTFT domain.

This leads to the following:

$x_c(t) = \cos 2\pi F_0 t$

$F_0 = 8$ kHz, $F_s = 24$ kHz

$x[n] = \cos 2\pi \frac{8\times10^3}{24\times10^3} n$

$= \cos \frac{2\pi n}{3}$

$y_c(t) = \cos 2\pi F_0 t$

$F_0 = 16$ kHz, $F_s = 48$ kHz

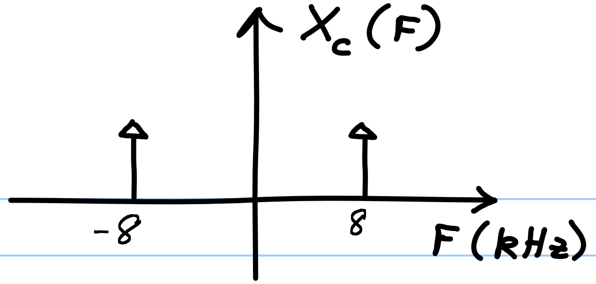
$y[n] = \cos 2\pi \frac{16\times10^3}{48\times10^3} n$

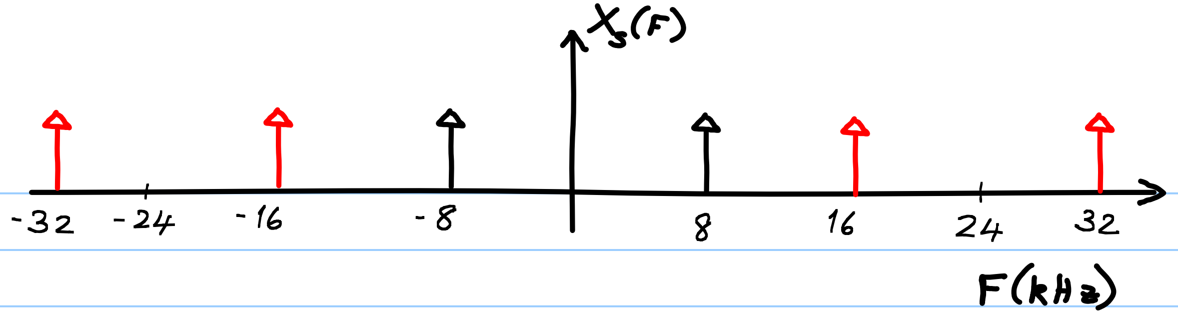
$= \cos \frac{2\pi n}{3}$

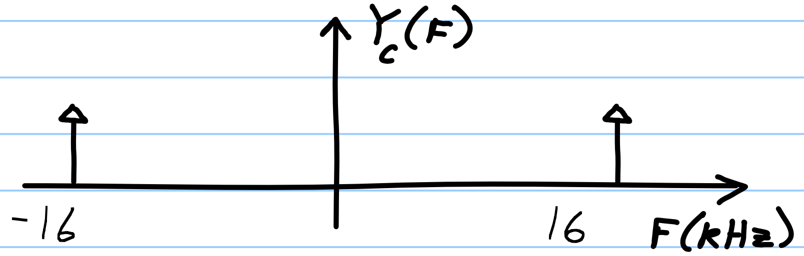
$= x[n]$

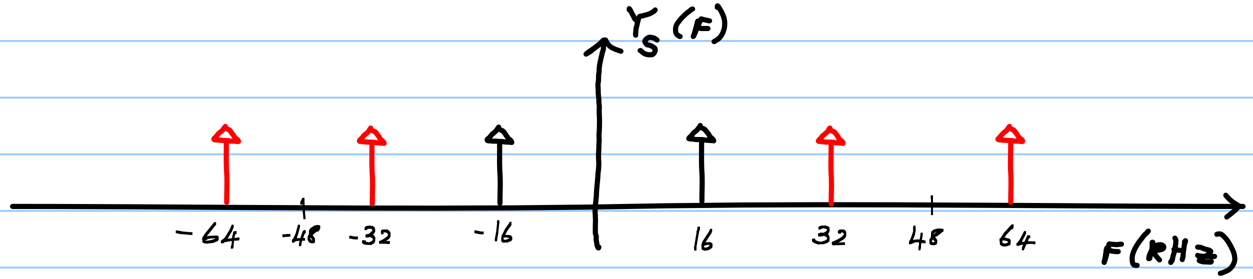
Thus, given $x[n] = \cos \frac{2\pi n}{3}$, one cannot tell whether it is a result of sampling an 8 kHz signal at 24 kHz or a 16 kHz signal at 48 kHz. To deduce the true signal frequency in Hz from the given sampled sequence, we need information about F_s

Note that $X_s(F)$ and $Y_s(F)$ have no ambiguity in revealing the true signal frequency.

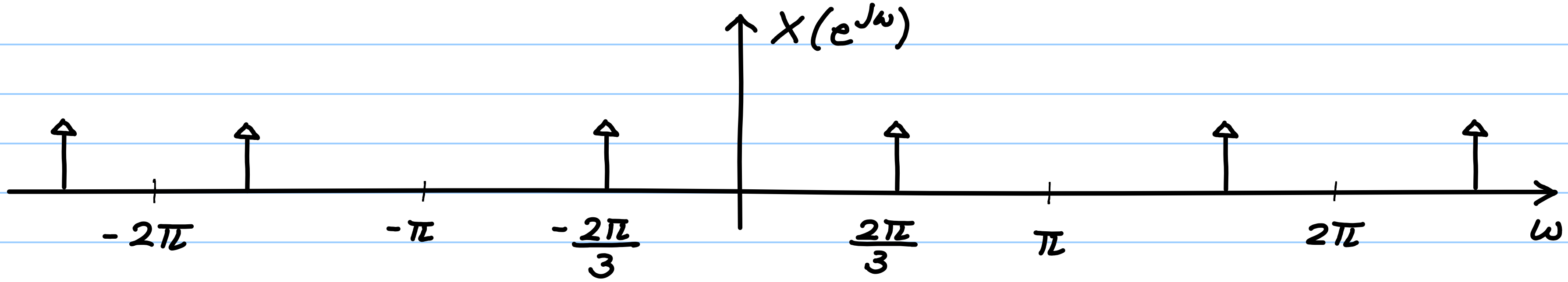








Both $X_s(F)$ and $Y_s(F)$ map to the same $X(e^{j\omega})$:



The Discrete Fourier Transform (DFT)

Recall the various Fourier representations we have seen so far:

Indep. Variable	Periodic ?	Spectrum	Periodic ?	
continuous	yes	line	no	CTFS
continuous	no	continuous	no	CTFT
discrete	no	continuous	yes	DTFT
discrete	yes	line	yes	DTFS

$$x(t+T) = x(t) \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} dt$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$x[n+N] = x[n] \qquad x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k n}{N}} \qquad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

Suppose $x[n]$ is known for $n = 0, 1, 2, \ldots, N-1$. We define the DFT as follows:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}}$$

From the definition it follows that $X[k+N] = X[k]$ and hence the range of 'k' of interest is $\boxed{k = 0, 1, 2, \ldots, N-1}$.

The DFT can be expressed using matrix-vector notation.

$$\underset{N\times N}{\underbrace{\begin{bmatrix} & \uparrow & \\ \longleftarrow & e^{-j\frac{2\pi k n}{N}} & \xrightarrow{} n \\ & \downarrow k & \end{bmatrix}}_{\underline{W}}} \underset{N\times 1}{\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\underline{x}}} = \underset{N\times 1}{\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\underline{X}}}$$

It can easily be verified that $\underline{W}$ is full rank, i.e., rank N, and hence invertible. Therefore $\underline{x}$ can be obtained from $\underline{X}$ as follows:

$$\underline{x} = \underline{W}^{-1}\underline{X}$$

In equation form, the above can be expressed as,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k n}{N}}$$

The inverse transform implies $x[n] = x[n+N]$. That is, even though no assumption was made about $x[n]$ outside $[0, N-1]$, the DFT framework imposes periodicity on $x[n]$.

Thus, both $x[n]$ and $X[k]$ are periodic. This is reminiscent of DTFS. In fact the DFT is nothing but a slightly modified version of the DTFS!

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}}$$

Thus,

$$\boxed{X[k] = Na_k}$$

<u>Exercise</u>

Show that $\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$ gives back $x[n]$.

DFT as the samples of the DTFT

Suppose we assume that x[n] is zero outside [0, N-1]

Then,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

If we sample $X(e^{j\omega})$ at N uniformly spaced points, i.e., at

$\omega_k = \frac{2\pi k}{N}$ for $k = 0, 1, 2, \ldots, N-1$, then

$$X(e^{j\omega})\Big|_{\omega=\omega_k} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n} = X[k]$$

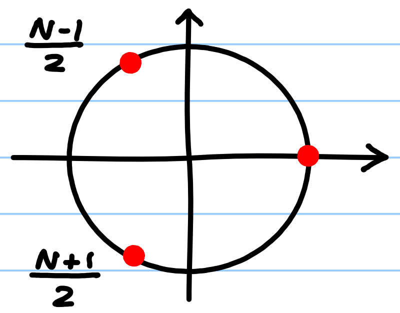
Thus, another interpretation of the DFT is viewing it as the samples of the DTFT.

Since $X(e^{j\omega_k}) = X[k]$, the inverse transform expression is identical, the implication of which is that we get $\tilde{x}[n]$ back, rather than $x[n]$. However, as before, $\tilde{x}[n] = x[n]$ for $0 \leq n \leq N-1$.

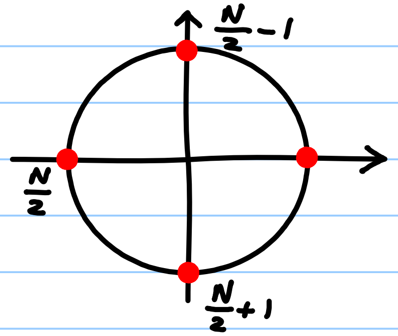
To interpret the periodicity, we see that $x[n]$ becomes $\tilde{x}[n]$ because of sampling $X(e^{j\omega})$. ["Sampling in one domain results in a periodic repetition in the other domain."]

If N is odd, there will not be a sample corresponding to $\omega = \pi$

If N is even, there will be a sample corresponding to $\omega = \pi$



$N = 3$



$N = 4$

Since both $x[n]$ and $X[k]$ are periodic, we need to consider the indices over the range $[0, N-1]$ only.

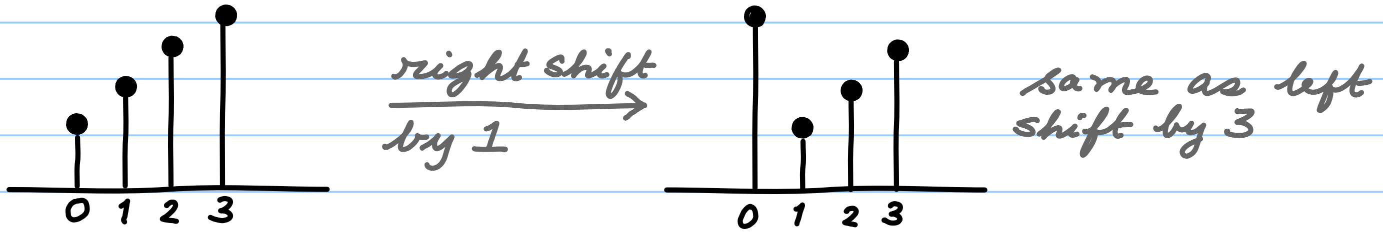
That is, the index 'l' is replaced by 'l mod N', and denoted by $\langle l \rangle_N \equiv l \bmod N$.

Properties

1) $a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1[k] + a_2 X_2[k]$

2) $x[n-n_0] \equiv x[\langle n-n_0 \rangle_N] \longleftrightarrow e^{-j\frac{2\pi k n_0}{N}} X[k]$

$x[\langle n-n_0 \rangle_N]$ is called as a circular shift.



Note that there is no relationship, in general, between the DTFT of $x[n]$ and $x[\langle n-n_0 \rangle_N]$. However, the corresponding DFTs share the relationship given above.

Also, since the sequence has the implied periodicity, a shift of $n-n_0$, where $0 \leq n_0 \leq N-1$, is the same as $n+m_0$ where $m_0 = N-n_0$.

3) $$e^{j\frac{2\pi \ell n}{N}} x[n] \longleftrightarrow X[k-\ell] \equiv X[\langle k-\ell \rangle_N]$$

4) $x[n] \circledast_N y[n] = \sum_{m=0}^{N-1} x[m]\, y[n-m] \longleftrightarrow X[k]\, Y[k]$

5) $x[n]\, y[n] \longleftrightarrow \frac{1}{N} X[k] \circledast_N Y[k]$

6) $\underline{x} = (x[0], x[1], \ldots, x[N-1])^T$

$$\|\underline{x}\|_2^2 = \underline{x}^H \underline{x}$$

$$\|\underline{X}\|_2^2 = \underline{X}^H \underline{X} = (\underline{W}\,\underline{x})^H (\underline{W}\,\underline{x})$$

$$= \underline{x}^H \underline{W}^H \underline{W}\, \underline{x}$$

$$= N\, \underline{x}^H \underline{x}$$

Hence, $\underline{x}^H \underline{x} = \frac{1}{N} \underline{X}^H \underline{X}$, i.e, $\|\underline{x}\|_2^2 = \frac{1}{N} \|\underline{X}\|_2^2$

7) $x^*[n] \longleftrightarrow X^*[-k] = X^*[N-k]$

$\Rightarrow$ $X[1] = X^*[N-1]$,
$X[2] = X^*[N-2]$, and so on.

Recovering the DTFT from the DFT

If the DTFT of an N-point sequence is sampled at a set of at least N uniformly spaced points, we can recover the DTFT from the DFT without any loss in information.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k n}{N}} \right] e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X[k]\, P\left(\omega - \frac{2\pi k}{N}\right)$$

where $P(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = e^{-j\omega (N-1)/2} \frac{\sin N\omega/2}{\sin \omega/2}$

Note that $P\left(\frac{2\pi k}{N}\right) = \begin{cases} 1 & k=0 \\ 0 & k=1,2,\ldots,N-1 \end{cases}$

Note Title 31-10-2013

EE 5330 Oct. 31, 2013

$X[k]$ = "k^{th} DFT bin value"

How can we translate bin index to true analog frequency?

$$X[k] = X[k+N]$$

$$X_s(F) = X_s(F+F_s)$$

Hence, the k^{th} bin maps to $\frac{k}{N} \cdot F_s$ (zero-based index)

$k = 0$ maps to 0 Hz

$k = 1$ maps to $\frac{F_s}{N}$ Hz

$\vdots$

$k = N-1$ maps to $\frac{N-1}{N} F_s$ Hz

Effects of Zero Padding

Consider the N-point sequence $x[n]$ (defined over $n = 0, 1, \ldots, N-1$) and its N-point transform $X[k]$.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} \qquad k = 0, 1, \ldots, N-1.$$

Now consider the following L-point ($L > N$) sequence $y[n]$:

$$y[n] = \begin{cases} x[n] & n = 0, 1, \ldots, N-1 \\ 0 & n = N, N+1, \ldots, L-1 \end{cases}$$

$y[n]$ is the zero-padded version of $x[n]$.

Consider the following L-point DFT of y[n]:

$$Y[k] = \sum_{n=0}^{L-1} y[n] e^{-j\frac{2\pi kn}{L}} \qquad k = 0, 1, \ldots, L-1$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{L}} \qquad k = 0, 1, \ldots, L-1$$

Note that $Y(e^{j\omega}) = \sum_{n=0}^{L-1} y[n] e^{-j\omega n}$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

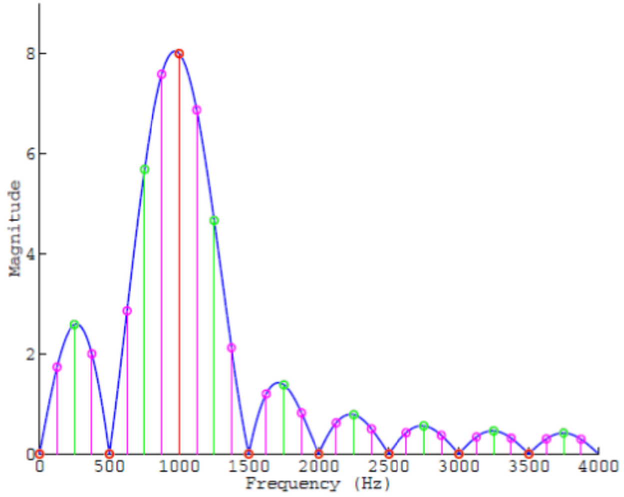
$$= X(e^{j\omega})$$

That is, the underlying DTFT remains the same.

However, since the DFT can be interpreted as sampling the DTFT, zero-padding enables us to sample the underlying DTFT at a finer set of points.

$x[n] = \sin \frac{2\pi n}{8} \quad n = 0, 1, \ldots, 15$

Blue: DTFT
Red: 16-point DFT
Green: 32-point DFT (contains the red samples as a subset)
Magenta: 64 point DFT (contains the red & green samples as a subset)



DTFT and 64-pt DFT

EE5330 Nov. 4, 2013

The Fast Fourier Transform (FFT) Algorithm:

Recall the following DFT definition, where the notation $W_N = e^{-j\frac{2\pi}{N}}$ is used. The suffix N in W_N is dropped when there is no ambiguity. It is convenient to use x_k instead of $x[k]$.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & & & & \\ 1 & W^{N-1} & & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

Each row involves N multiplies (neglecting the fact that this is not true for the first row and first column). Since there are N such rows, the no. of multiplies is $N \cdot N$, or N^2. Also, for each X_k, we require $N-1$ additions, and totally there are $N \cdot (N-1)$ additions, or, roughly N^2 additions. Thus, the straightforward computation of the DFT requires N^2 multiplications and N^2 additions approximately.

Decimation-in-Time FFT Algorithm:

Consider breaking up $\underline{x}$ into its odd & even indices before computing the DFT. That is

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi k n}{N}} = \sum_{n=0}^{N-1} x_k W_N^{kn} \qquad k = 0, 1, \dots, N-1.$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x_{2r} e^{-j\frac{2\pi k}{N}2r} + \sum_{r=0}^{\frac{N}{2}-1} x_{2r+1} e^{-j\frac{2\pi k}{N}(2r+1)}$$

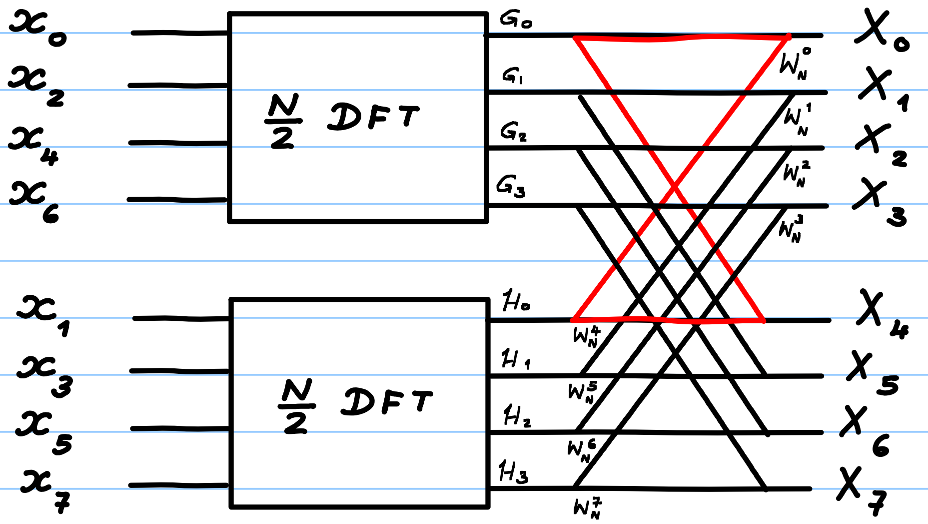
$$= \sum_{r=0}^{\frac{N}{2}-1} g_r e^{-j\frac{2\pi k r}{N/2}} + e^{-j\frac{2\pi k}{N}} \sum_{r=0}^{\frac{N}{2}-1} h_r e^{-j\frac{2\pi k r}{N/2}}$$

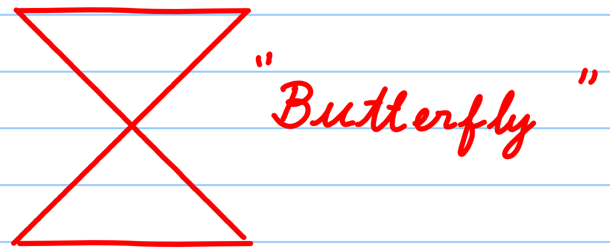
$$= \sum_{r=0}^{\frac{N}{2}-1} g_r W_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} h_r W_{\frac{N}{2}}^{rk}$$

These are two $\frac{N}{2}$-point DFTs !

$$= G_k + W_N^k H_k \qquad k = 0, 1, 2, \dots, N-1$$

Since $\{G_k\}_{k=0}^{\frac{N}{2}-1}$ & $\{H_k\}_{k=0}^{\frac{N}{2}-1}$ are $\frac{N}{2}$-point DFTs, they are periodic with period $\frac{N}{2}$. Thus $G_{\frac{N}{2}} = G_0$, and so on. The above set of equations can be represented using the following block diagram.





Recall that the no. of mult./adds for an N-point transform is N^2.

The computation derived above has two $\frac{N}{2}$ transforms. They require $2 \cdot \left(\frac{N}{2}\right)^2$ multiplies. In addition, combining them via W_N^k requires N multiplies. Thus, $N^2 \longrightarrow 2 \cdot \left(\frac{N}{2}\right)^2 + N$ is the reduction in the no. of multiplies.

<u>Example</u>

If $N = 8$, $N^2 = 64$, whereas $2 \cdot \left(\frac{N}{2}\right)^2 + N = 2 \cdot 16 + 8 = 40 < 64$. Thus, this "divide and conquer" approach leads to computational savings.

The sequences $\underline{g}$ and $\underline{h}$ can further divided into their odd and even indices and the above strategy can be exploited once more.

If N is a power of 2, the division by two can be carried out $\log_2 N$ times.

The computational savings follow the same pattern:

$$N^2 \longrightarrow 2\underbrace{\left(\frac{N}{2}\right)^2} + N \longrightarrow 2\left[2\left(\frac{N}{4}\right)^2 + \frac{N}{2}\right] + N$$

$$= 4\left(\frac{N}{4}\right)^2 + 2N \quad \text{second stage}$$

For $N = 8$, we get $64 \to 40 \to 32$

In general, if N is a power of 2, we can continue to divide by two until we reach segments of length two. The no. of such segments is $\log_2 N$.

$$N^2 \longrightarrow 2\left(\frac{N}{2}\right)^2 + N \longrightarrow 4\left(\frac{N}{4}\right)^2 + N + N \longrightarrow 4\left[2\left(\frac{N}{8}\right)^2 + \frac{N}{4}\right] + N + N$$

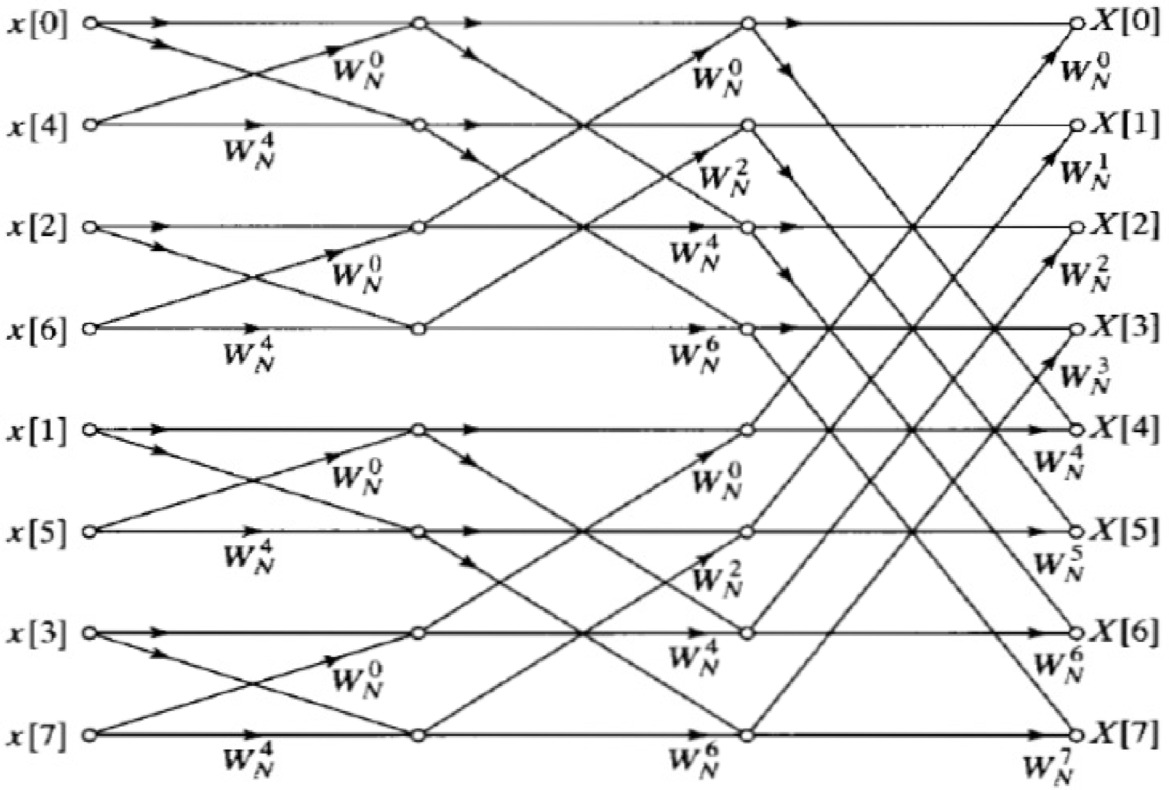
$$= 8\left(\frac{N}{8}\right)^2 + 3N \quad \text{third stage}$$

Thus, if there are $\log_2 N$ segments, the no. mults./adds will be $\boxed{N \cdot \log_2 N}$ approximately.

Thus, from $O(N^2)$ mult./adds, we have come to $N \log_2 N$ mult./adds. If $N = 1024$, the savings is hundred-fold, i.e., two orders of magnitude!

The FFT algorithm is the principal reason why DSP is as practical and as powerful as it has become today. But for the presence of such FFT-type algorithms, DSP would have remained only as an academic curiosity!

Butterfly Diagram for an 8-pt FFT (DIT Algorithm):



From "Discrete-Time Signal Processing"
by Oppenheim & Schafer

For $N = 8$, the order in which the i/p appears is,

$$x_0 \quad x_2 \quad x_4 \quad x_6 \quad x_1 \quad x_3 \quad x_5 \quad x_7$$

$$\downarrow$$

$$g_0 \quad g_1 \quad g_2 \quad g_3 \quad h_0 \quad h_1 \quad h_2 \quad h_3$$

$$\downarrow$$

$$g_0 \quad g_2 \quad g_1 \quad g_3 \quad h_0 \quad h_2 \quad h_1 \quad h_3$$

$$\downarrow$$

$$x_0 \quad x_4 \quad x_2 \quad x_6 \quad x_1 \quad x_5 \quad x_3 \quad x_7$$

This order is nothing but the one obtained by (i) representing the index in binary form, and (ii) bit reversing the representation.

index :	0	1	2	3	4	5	6	7
binary Form :	000	001	010	011	100	101	110	111
bit reversal :	000	100	010	110	001	101	011	111
bit reversed index :	0	4	2	6	1	5	3	7

Note that the final stage consists of 2-point DFTs.

If $\{p_0, p_1\}$ is the two point sequence, then,

$$P_0 = p_0 + p_1$$

$$P_1 = p_0 + p_1 e^{-j\pi} = p_0 - p_1$$

Similar to the Decimation in Time (DIT) algorithm, there exists the Decimation in Frequency (DIF) algorithm, wherein computational savings are obtained by dividing the sequence into its first and second halves successively (rather than into its odd and even indices).

We get the same computational savings, but the X_k now appear in bit reversed order.

For N that is not a power of 2, FFT algorithms exist by factoring N into its prime factors (resulting in the so-called Prime Factor Algorithm).

When $N = 2^{\gamma}$, such FFT algorithms are called radix 2 algorithms.

Efficient algorithms exist even when N is prime!

Since the structure of the Inverse DFT (IDFT) is fundamentally similar to that of the DFT, FFT algorithms can be applied for the efficient computation of the IDFT also with only trivial modifications.

Because of the existence of the FFT class of algorithms, for $N > 60$ (roughly), it is more efficient to realize convolution of two sequences by multiplying their respective transforms and computing the inverse transform of the product.