EE5330: Digital Signal Processing

Tutorial 6

(October 24, 2013)

1. Let $x[n]$ be a real-valued input to a causal and stable allpass system and $y[n]$ be its output. $x[n] = 0$ for $n < 0$. We wish to show that for any n_0 the following inequality holds:

$$\sum_{n=0}^{n_0} x^2[n] \geq \sum_{n=0}^{n_0} y^2[n]$$

   (a) First form the sequence $x_1[n] = x[n]$ for $n \leq n_0$ and zero for $n > n_0$. Let $y_1[n]$ be the corresponding response of the allpass system. Given that the system is causal, what is the relationship between $y[n]$ and $y_1[n]$ for $n \leq n_0$?

   (b) Express $\sum_{n=0}^{n_0} x^2[n]$ in terms of an infinite sum involving $x_1[n]$. Now relate this to an expression involving $y_1[n]$ and hence deduce $\sum_{n=0}^{n_0} x^2[n] \geq \sum_{n=0}^{n_0} y^2[n]$.

2. Let $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$, where both H_1 and H_2 are rational transfer functions. It is further given that H_1 is *minimum phase*. Let $y_1[n]$ and $y_2[n]$ be the output of the two systems for a given input $x[n]$. Recall the decomposition $H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$ and use the result of the previous problem to show that

$$\sum_{n=0}^{n_0} y_1^2[n] \geq \sum_{n=0}^{n_0} y_2^2[n]$$

3. Consider the following $H(z)$:

$$H(z) = \prod_{k=1}^{p} \frac{-z_k^* + z^{-1}}{1 - z_k z^{-1}} \qquad z_k = r_k e^{j\omega_k}, \;\; 0 < r_k < 1$$

   Establish the following:

$$|H(z)| = \begin{cases} > 1 & |z| < 1 \\ < 1 & |z| > 1 \\ = 1 & |z| = 1 \end{cases}$$

   which is an important property of an allpass filter.

4. Let $h_{\text{min}}[n]$ denote a minimum-phase sequence with z-transform $H_{\text{min}}(z)$. If $h[n]$ is a causal non-minimum-phase sequence whose Fourier transform magnitude is equal to $\left|H_{\text{min}}(e^{j\omega})\right|$, show that $|h[0] < |h_{\text{min}}[0]|$. Use the initial-value theorem together with the fact that any $H(z)$ can be factored as $H_{\text{min}}(z)H_{\text{ap}}(z)$. As a first step, let $H_{\text{ap}}(z)$ be of first order. The general case easily follows because an m-th order allpass filter can be thought of as a product of m first order stages.

5. Consider a causal sequence $x[n]$ with z-transform

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z)}{(1-\frac{1}{6}z)}$$

For what values of α is $\alpha^n\, x[n]$ a real, minimum-phase sequence?

6. Determine the group delay for $0 < \omega < \pi$ for each of the following sequences:

(a)

$$x_1[n] = \begin{cases} n-1 & 1 \le n \le 5 \\ 9-n & 5 < n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$x_2[n] = \left(\frac{1}{2}\right)^{|n-1|} + \left(\frac{1}{2}\right)^{|n|}$$

7. The frequency response of any rational transfer function $H(z)$ can be expressed as $A(\omega) \cdot e^{j\phi(\omega)}$, where $A(\omega)$ is a real-valued function (i.e., it can take on negative values as well) and $\phi(\omega)$ is *continuous* (i.e., it does not have any jumps—jumps of π cause $A(\omega)$ to go negative). $A(\omega)$ is also called the *zero phase response*. In this and subsequent problems, the term "zero phase response" will refer to $A(\omega)$ (the notation $H_{ZP}(\omega)$ is also used).

Let $H(z)$ be a Type-I linear phase LPF of length N $(= M+1)$, for which $H_{ZP}(\omega) \approx 1$ for $0 \le \omega \le \frac{\pi}{4}$ and $H_{ZP}(\omega) \approx 0$ for $\frac{\pi}{3} \le \omega \le \pi$. Another filter $G(z)$ is defined as $G(z) = z^{-M/2} - H(z)$.

(a) Express the impulse response $g[n]$ in terms of $h[n]$. Is $g[n]$ also a Type-I filter?

(b) Obtain an expression for $G_{ZP}(\omega)$ in terms of $H_{ZP}(\omega)$. Make a rough sketch of $G_{ZP}(\omega)$ for $0 \le \omega \le \pi$.

(c) Compare the behaviour of $G_{ZP}(\omega)$ with that of $F_{ZP}(\omega)$, where $F(z) = (-1)^{M/2}H(-z)$.

8. Let $H(z)$ be a Type-I or Type-II lowpass filter of length N with corresponding $H_{ZP}(\omega)$. Define a new linear phase filter $g[n] = 2\cos[\omega_0(n - M/2)] \cdot h[n]$.

(a) Express the frequency response $G(e^{j\omega})$ in terms of $H(e^{j\omega})$, and thereby $G_{ZP}(\omega)$ in terms of $H_{ZP}(\omega)$.

(b) Let $H(z)$ be a Type-II filter with $\omega_0 = \pi/2$. Assuming the plot of $H_{ZP}(\omega)$ over $0 \le \omega < 2\pi$, sketch the corresponding $G_{ZP}(\omega)$ over the same interval.

9. Let $H(z)$ be an arbitrary Type-I or Type-II linear phase filter filter of length N. Define $G(z) = H(z^3)$.

(a) Express $g[n]$ in terms of $h[n]$.

(b) Show that $g[n]$ is also a linear phase filter, and determine its length and type. Express $G_{ZP}(\omega)$ in terms of $H_{ZP}(\omega)$.

(c) Let $H_{\text{ZP}}(\omega) = 2(\cos\omega - \cos\theta)$. Determine $G_{\text{ZP}}(\omega)$ and its zero-crossings in $[0, \pi]$ in terms of θ. Assume $0 < \theta < \pi$.

10. Design a Type-III linear phase filter $H(z)$ with $M = 4$ satisfying the following conditions: $H_{\text{ZP}}(0.3\pi) = 0.2$ and $H_{\text{ZP}}(0.6\pi) = 0.8$.

11. The transfer function $H(z)$ of a Type-IV filter with $M = 7$ has zeros at $0.8\, e^{j\pi/4}$ and at $z = -2$. Determine *all* the zeros of $H(z)$ assuming a real-valued impulse response.

12. Given $F_1(z) = 2.1 - 3.5z^{-1} + 4.2z^{-2}$, we wish to form a linear-phase filter $H(z)$ that satisfies $H(z) = F_1(z) \cdot F_2(z)$. Determine the *lowest order* $F_2(z)$ needed to accomplish this. The impulse response $h[n]$ has to be real-valued.

13. Consider he cascade of the Type-II linear phase filter $H_2(z)$ of length $M_2 + 1$ and the Type-III filter $H_3(z)$ of length $M_3 + 1$. Show that the cascade represents another linear phase filter, and determine its length and type. Express the zero-phase response of the cascade in terms of the individual zero-phase responses. Repeat this exercise for as many pairs as you fancy.

14. The ideal Type-IV differentiator is an LTI system with frequency response

$$H_d(e^{j\omega}) = j\omega\, e^{-j\omega M/2} \qquad -\pi < \omega \le \pi$$

where M is an odd integer.

(a) Derive the impulse response $h_d[n]$ of this ideal system in terms of M. Show that $h_d[n] = -h_d[M - n]$.

(b) Obtain a Type-IV FIR filter $h[n]$ by truncating $h_d[n]$ to the interval $0 \le n \le M$. For $M = 5$, compute and plot the resulting $H_{\text{ZP}}(\omega)$ for $0 \le \omega \le \pi$.

15. **Computer assignment** Execute the commands `freqz([1,a],1,4000)` and `freqz([a, 1], 1, 4000)` and observe carefully the similarities and differences between the two results for `a = 0.99`. Vary `a` and observe the effects on the magnitude and phase responses. What happens when `a = 1`?

16. **Computer assignment** Plot the magnitude and phase response of a filter with a first order zero on the unit circle and compare it with that of a filter with a second order zero at the same location.

17. **Computer assignment** Plot the unwrapped phase angle for a stable all-pass filter. Verify its magnitude response is unity for all frequencies. Try various orders and pole locations (once you choose the poles, the zeros are automatically fixed). Does the phase function exhibit any particular characteristic? What are the values of the phase at 0, π, and 2π? Is the phase angle at $\omega = 2\pi$ related to the filter order?

18. **Computer assignment** An FIR filter using the Parks-McClellan algorithm can be designed using the command `h = firpm(32, [0 0.3 0.4 1], [1 1 0 0])`. Let `H = freqz(h, 1, 4000)`. Plot magnitude in both linear and log scales and observe the differences. Plot the unwrapped phase angle and observe its properties. Plot `zplane(`

`roots(h)`) and verify that jumps of π occur in the phase response at unit circle zeros locations. Let $x[n] = \cos(\omega_1 n)$, where $\omega_1/2\pi = 0.1$. Filter this sequence through the above filter by convolving the input with the impulse response. By suitably delaying the input are you able to align it exactly with the output? Ignore the transients at the beginning and at the end. Is this delay related to the length of the filter? Is it related to the slope of the phase response? Now reduce the length of the filter by one by changing the argument of `firpm`. Are you now able to align exactly the input and the filtered output? What is the slope of the phase response? Explain your answer.

19. **Computer assignment** Let us compare how an IIR filter with almost identical magnitude response, but whose phase is highly nonlinear, behaves when an input with two tones is applied to it. An *elliptic filter* can be designed using the command `[b,a] = ellip(5,0.3,35,0.3)`. Compare the magnitude response of this filter with the one designed in the previous example. Let us consider the response to $x[n] = \cos(\omega_1 n) + \cos(\omega_2 n)$, where ω_1 is as before and $\omega_2/2\pi = 0.15$. Note that the second component is at the band edge, where the magnitude is still close to unity for both filters, but the phase response changes rapidly for the elliptic filter. Plot the group delay using `grpdelay(b, a, 4000)`. How different are the values for the two tones? Compute the output using `filter(b, a, x)` and plot the result. Is it possible to align the elliptic filter's input and output by suitably delaying one of them? Carefully contrast this filter's behaviour (nonlinear phase response) with that of its counterpart in the previous problem (exactly linear phase response). Add a third tone with frequency $\omega_3/2\pi = 0.4$ to the input and verify that both filters attenuate it significantly, illustrating frequency selective filtering. You can also add noise and see how the components that fall in the stopband get attenuated.

20. **Computer assignment** Using MATLAB try problems M 7.3, M 7.5, M 7.6 on pp. 424–425 of S.K. Mitra's "Digital Signal Processing" (Third Edition, Tata McGraw-Hill, 2006).

21. **Computer assignment** Design a first-order all-pole lowpass filter with 3-dB cutoff $\omega_c = 0.25\pi$. Ensure unity gain at $\omega = 0$.

22. **Computer assignment** Consider the ideal lowpass filter with cutoff frequency ω_c. Its impulse response is $h[n] = \dfrac{\sin \omega_c n}{\pi n}$. Let $\omega_c = 0.25\pi$. Truncate $h[n]$ by multiplying it with a rectangular window $w[n]$ over the interval $-N \leq n \leq N$. Now make $h_1[n] = h[n] \cdot w[n]$ causal by shifting it by N samples. Plot the magnitude and phase of $H_1(e^{j\omega})$ using the command `freqz(h1,1,4000)`. Normalize the filter such that the gain at $\omega = 0$ is unity. What is the filter's peak sidelobe level (in dB) for $N = 10$, 20, and 100 ? Plot also the log magnitude spectrum of the rectangular window for the values of N given above. Ensure that the peak gain is unity. For the window transform, what is the peak sidelobe level (in dB) for the values of N given above ? ◇