EE5330: Digital Signal Processing

Tutorial 5

(October 3, 2013)

1. Consider the improved bandpass filter with zeros at $z = \pm 1$. Its transfer function can be expressed in the following form:

$$H_{\text{BP}}(z) = \frac{1-\alpha}{2} \cdot \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

Note that the poles are at $re^{\pm j\theta}$, which means that $r = \sqrt{\alpha}$ and $2\sqrt{\alpha}\cos\theta = \beta(1+\alpha)$.

   (a) Show that magnitude transfer function always has a peak at $\omega_0 = \cos^{-1}(\beta)$. This is in contrast to the BPF that has no zeros at $z = \pm 1$ (which has a peak at $\cos^{-1}\left(\frac{1+r^2}{2r}\cos\theta\right)$ only if $-1 \le \frac{1+r^2}{2r}\cos\theta \le 1$). Rather than trying to find the peak location by setting the derivative of the magnitude-squared transfer to zero, express $|H_{\text{BP}}(e^{j\omega})|^2$ in the form $x^2/(x^2+y^2)$ and show that the peak occurs when $y = 0$, which requires $\cos\omega$ equals β. This value of ω is denoted by ω_0. Observe that $|H_{\text{BP}}(e^{j\omega_0})|^2 = 1$.

   (b) Let ω_{c_1} and ω_{c_2} be such that $\omega_{c_1} < \omega_0 < \omega_{c_2}$ and $|H(e^{j\omega_{c_1}})|^2 = |H(e^{j\omega_{c_2}})|^2 = \frac{1}{2}$. Let $B_w = \omega_{c_2} - \omega_{c_1}$ be the 3-dB bandwidth. Show that $B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$.

2. Let $H(z) = (a^* + z^{-1})/(1 + az^{-1})$ be a first order stable all-pass filter. Note that $H(e^{j\omega})$ is of the form $\exp[j\phi(\omega)]$. Write the expression for $\phi(\omega)$ and $\phi'(\omega)$. Deduce that $\phi(\omega)$ is a monotone decreasing function in the range $[0, 2\pi)$. In particular, for real-valued a, what values does it take at 0, π, and 2π? Geometrically also it can be seen that $\phi(\omega)$ is monotone decreasing: Let P be the point at which the pole is located; O be the point on the unit circle; and Z be location of the zero. Consider the variation of the angle of POZ as the point O moves along the unit circle from 0 to 2π. From this deduce that the change in angle is -2π.

3. Let

$$H(z) = \frac{(z-1)^2}{z^2 - 1.212436\,z + 0.49}$$

Is $H(z)$ lowpass, highpass, bandpass, or bandstop?

4. Let $x[n] = s[n]\cos(\omega_0 n)$ be input to a filter with transfer function $H(e^{j\omega})$. $s[n]$ is lowpass and very narrowband, i.e., $S(e^{j\omega}) = 0$ for $|\omega| > \Delta$, with Δ very small and $\Delta \ll \omega_0$, so that $X(e^{j\omega})$ is narrowband around $\omega = \pm\omega_0$; in this region $|H(e^{j\omega})| \approx 1$.

   (a) If $\sphericalangle H(e^{j\omega}) = -\phi_0 \operatorname{sgn}(\omega)$ show that $y[n] = s[n]\cos(\omega_0 n - \phi_0)$.

(b) If $\sphericalangle H(e^{j\omega})$ is linear with a slope of $-n_d$ with $\sphericalangle H(e^{j0^+}) = -\phi_0$ (the overall phase function is odd-symmetric), then show that $y[n] = s[n - n_d]\cos(\omega_0 n - \phi_0 - \omega_0 n_d) = s[n - n_d]\cos(\omega_0 n - \phi_1)$, where $\phi_1 = \sphericalangle H(e^{j\omega_0})$.

(c) Group delay is defined as the negative of the derivative of the unwrapped angle (continuous phase function). The *phase delay* is defined as $\tau_{\text{ph}}(\omega) = (-1/\omega)\sphericalangle H(e^{j\omega})$. Based on the previous results, show that $y[n] = s[n - \tau_{\text{gr}}(\omega_0)]\cos\{\omega_0[n - \tau_{\text{ph}}(\omega_0)]\}$.

Hint: How are $X(e^{j\omega})$ and $S(e^{j\omega})$ related? Since $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, use the corresponding equations for $H(e^{j\omega})$ and simplify.

5. The frequency response of any rational transfer function $H(z)$ can be expressed as $A(\omega) \cdot e^{j\phi(\omega)}$, where $A(\omega)$ is a real-valued function (i.e., it can take on negative values as well) and $\phi(\omega)$ is *continuous* (i.e., it does not have any jumps—jumps of π cause $A(\omega)$ to go negative). $A(\omega)$ is also called the *zero phase response*. In this and subsequent problems, the term "zero phase response" will refer to $A(\omega)$ (the notation $H_{\text{ZP}}(\omega)$ is also used).

   Derive a closed-form expression for $H_{\text{ZP}}(\omega)$ for (a) $H(z) = (1 + z^{-1})^2(1 - z^{-1})$, (b) $H(z) = (1 + z^{-2})(1 - z^{-1})^2$, and (c) $H(z) = 1 - 1.5z^{-1} + 0.5z^{-2}$ For (a), (b), and (c) derive and sketch the unwrapped phase over the interval $0 \le \omega \le \pi$.

6. The transfer function of five FIR filters with identical magnitude responses are given below:

$$
\begin{aligned}
H_1(z) &= 1 - 0.5z^{-1} + 0.8z^{-2} - 0.4z^{-3} + 0.25z^{-4} - 0.125z^{-5} + 0.2z^{-6} - 0.1z^{-7} \\
H_2(z) &= 0.5 + 0.25z^{-1} + 0.4z^{-2} - 0.425z^{-3} + 0.75z^{-4} - 0.75z^{-5} + 0.6z^{-6} - 0.2z^{-7} \\
H_3(z) &= -0.25 + 0.25z^{-1} + 0.175z^{-2} + 0.7z^{-3} - 0.45z^{-4} + 0.9z^{-5} - 0.6z^{-6} + 0.4z^{-7} \\
H_4(z) &= -0.5 + z^{-1} - 0.4z^{-2} + 0.8z^{-3} - 0.125z^{-4} + 0.25z^{-5} - 0.1z^{-6} + 0.2z^{-7} \\
H_5(z) &= -0.1 + 0.2z^{-1} - 0.125z^{-2} + 0.25z^{-3} - 0.4z^{-4} + 0.8z^{-5} - 0.5z^{-6} + z^{-7}
\end{aligned}
$$

   Which transfer function has all its zeros outside the unit circle? Which transfer function has all its zeros inside the unit circle? If possible, specify at least one other length-8 FIR filter having the same magnitude response as that of the above transfer functions. A simple approach would be to compute the roots of $H_i(z)$ using MATLAB. ◇