EE515: Mathematical Methods and Algorithms for Signal Processing

Assignment 3

(September 26, 2005)

1. 3.5–2, 3.8–3, 3.8–10.
2. 3.9–12, 3.11–17, 3.17–33.
3. 4.5–29, 4.6–33, 4.6–34.
4. The traditional Gram-Schmidt is not necessarily a good way computationally. Let ϵ be a number such that $1+\epsilon^2 \approx 1$ but $2+\epsilon$ and $\epsilon+\epsilon^2$ are computed fairly accurately. $\epsilon = 10^{-6}$ is typical on an 8- to 10-digit calculator, while $\epsilon = 10^{-10}$ is typical on a computer with maths coprocessor. Apply Gram-Schmidt to the following set of vectors:

$$\begin{bmatrix} 1 \\ 1+\epsilon \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1+\epsilon \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1+\epsilon \end{bmatrix}.$$

   The conditions on ϵ mean that the inner product of any different two of these will be accurately computed as $4+2\epsilon$, but the norm squared of each will be computed as $4+2\epsilon$, rather than $4+2\epsilon+\epsilon^2$. Find the angles between $\mathbf{u}_i$ and $\mathbf{u}_j$, $i,j = 1,2,3$ and $i \neq j$. Is any angle amiss?

5. The *modified Gram-Schmidt* method avoids the difficulties by modifying at the i-th stage *all* remaining $\mathbf{v}_i$, rather than just $\mathbf{v}_i$. In the old method, $\mathbf{v}_i$ is changed to $\mathbf{u}_i$ that is orthogonal to all the preceding $\mathbf{u}_j$. In the modified process, at the i-th step, *all* of the remaining $\mathbf{v}_i$ are altered so as to be made orthogonal to the most recently computed $\mathbf{u}$-vector. That is, starting with the given $\mathbf{v}_1, \ldots, \mathbf{v}_q$, define $\mathbf{v}_1^0 = \mathbf{v}_1, \ldots, \mathbf{v}_q^0 = \mathbf{v}_q$. At the i-th step, for $i = 1, \ldots, q$:

$$\begin{aligned} \mathbf{u}_i &:= \mathbf{v}_i^{i-1} \\ \mathbf{v}_j^i &:= \mathbf{v}_j^{i-1} - \frac{\langle \mathbf{u}_i, \mathbf{v}_j^{i-1} \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \qquad j = i+1, \ldots, q \end{aligned}$$

   Apply the modified method to the previous problem and recompute the angles. Comment on the result.

6. As before, $1+\epsilon$ is accurately computed but not $1+\epsilon^2$ (which evaluates to 1). Apply the traditional QR-decomposition to verify:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix} = \mathbf{Q}_0\mathbf{R}_0 = \begin{bmatrix} 1 & -\frac{\epsilon}{3} \\ 1 & -\frac{\epsilon}{3} \\ 1 & \frac{2\epsilon}{3} \end{bmatrix} \begin{bmatrix} 1 & 1+\frac{\epsilon}{3} \\ 0 & 1 \end{bmatrix}$$

7. Consider the matrix $\mathbf{A}$ in the previous problem and the least-squares solution of $\mathbf{Ax} \approx \mathbf{y}$ where $\mathbf{y} = [2\ \ 3\ \ 2]^T$. The true LS solution is

$$\mathbf{x} = \frac{1}{\epsilon} \begin{bmatrix} \frac{1}{2} + \frac{5\epsilon}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

   (a) Remembering that $3+2\epsilon+\epsilon^2$ evaluates to $3+2\epsilon$ find what will be *computed* as the *equations* $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{y}$ for finding $\mathbf{x}$.

   (b) Show that the unique solution to this computed system is

$$\frac{1}{\epsilon} \begin{bmatrix} -1+2\epsilon \\ 1 \end{bmatrix}$$

   which is very far from the correct LS solution just given above.

8. In an earlier problem you obtained the *computed* QR-decomposition for $\mathbf{A}$. Use this computed decomposition and solve the previous LS problem. Compare it with the true solution and that obtained using $\mathbf{A}^T\mathbf{A}$.

For grading, submit 3.8–3, 3.9–12, Q.4, and Q.5