EC5142: Introduction to DSP

Problem Set 6

(November 9, 2011)

1. Consider the z-transform pair denoted using the notation $x[n] \overset{Z}{\longleftrightarrow} X(z)$, where $r_1 < |z| < r_2$. Let $x_e[n]$ be the conjugate symmetric part of $x[n]$, i.e., $x_e[n] = x_e^*[-n]$; let $x_o[n]$ be the conjugate anti-symmetric part of $x[n]$, i.e., $x_o[n] = -x_o^*[-n]$. Similarly, let $X_e(z)$ and $X_o(z)$ be the conjugate symmetric and and anti-symmetric parts of $X(z)$, i.e., $X_e(z) = X_e^*(z^*)$ and $X_o(z) = -X_o^*(z^*)$. **Important Note**: Notationwise, $X_e(z)$ and $X_o(z)$ are *not* the z-transforms of $x_e[n]$ and $x_o[n]$.
   (a) Determine the z-transforms of $\mathrm{Re}\{x[n]\}$, $\mathrm{Im}\{x[n]\}$, $x_e[n]$, and $x_o[n]$.
   (b) Specialize the above results for $z = e^{j\omega}$, simplifying the result where possible.
   (c) Specialize the results in (b) when $x[n]$ is real-valued.

2. Determine the z-transform for each of the following sequences. Sketch the pole-zero plot and indicate the RoC. Also indicate whether or not the discrete-time Fourier transform exists, and if it does, whether it can be obtained by evaluating the z-transform on the unit circle. (a) $\delta[n]$, (b) $\delta[-n]$, (c) $\delta[n-1]$, (d) $\delta[n+1]$, (e) $-(1/2)^n u[-n-1]$, (f) $(1/2)^n u[-n]$, (g) 1, (h) $7 \cdot (1/3)^n \cos(2\pi n/6 + \pi/4) u[n]$, (i) $7 \cdot (1/3)^n \cos(2\pi n/6 + \pi/4)$, (j) a^n.

3. Let $x_1[n] = \alpha^n u[n]$ and $x_1[n] = \beta^n u[n]$ where $0 < \alpha, \beta < 1$ and $\alpha \neq \beta$. Find the $y[n] = x_1[n] * x_2[n]$ by computing the inverse z-transform of $X_1(z)X_2(z)$. In $y[n]$, if you let $\beta \longrightarrow \alpha$, does it tend to the inverse z-transform of $1/(1-\alpha z^{-1})^2$ (with RoC $|z| > |a|$) ?

4. Consider the finite-duration sequence $h[n]$ defined in the range $0 \leq n \leq N-1$. Let $g[n] = h^*[N-1-n]$.
   (a) Express $G(z)$ in terms of $H(z)$.
   (b) Now let $g[n] = h[n]$, i.e., $h^*[N-1-n] = h[n]$. If z_0 is a zero of $H(z)$, can you find another zero related to z_0 ?
   (c) Suppose now $h[n]$ is real-valued and z_0 is a non-trivial zero that is not located on the unit circle. How many other zeros of $H(z)$ can you determine from the knowledge of z_0 ?

5. The *autocorrelation* sequence corresponding to $x[n]$ is denoted by $r_{xx}[k]$ and computed using the formula $x[k] * x^*[-k]$. Express the z-transform of $r_{xx}[k]$ in terms of $X(z)$. How are the poles and zeros of the autocorrelation sequence related to those of $X(z)$? Ponder about the continuous-time counterpart of this property.

6. The transfer function $H(z)$ of an LTI system can be determined to within a scale factor given its pole/zero locations. Let the zeros be at c_1, c_2, and 0, with the poles at d_i, $i = 1, 2, 3$. Write the expression for $H(z)$ with factors of the form (i) $(z-a)$, and (ii) $(1-$

az^{-1}). Next, suppose the zeros are at c_1 and c_2, and the poles remain unchanged. Write down $H(z)$ in both forms. Note carefully the difference between these two examples. It is customary to use powers of z^{-1} in DSP literature because they correspond to causal delay elements. Note that the vector `c` in the expression `roots(c)` corresponds to the roots of the polynomial `C(1)*X^N + ··· + C(N)*X + C(N+1)`, i.e., the first coefficient corresponds to the highest power of `X`. In MATLAB be careful about whether a coefficient vector corresponds to a polynomial in z or z^{-1}.

7. Determine the RoC of the z-transform of the following sequences without explicitly finding $X(z)$; in each case specify whether or not the DTFT exists.
   (a) $x[n] = \left[\left(\frac{1}{2}\right)^n - \left(-\frac{4}{3}\right)^n\right] u[n-10]$
   (b) $x[n] = \left(-\frac{10}{9}\right)^n u[-n+4]$
   (c) $x[n] = \left(\frac{e}{\pi}\right)^n u[n+4]$
   (d) $x[n] = \left(\frac{e}{-\pi}\right)^n u[n] + (2+3j)^{n-2}\, u[-n-1]$

8. Determine $X(z)$ and the corresponding RoC for $x[n] = |n|a^{|n|}$ where $|a| < 1$.

9. Consider the rational function $X(z) = \dfrac{1+z^{-1}}{1+z^{-1}+z^{-2}}$. Expand $X(z)$ into a power series such that the series is convergent at $z = 0$. Identify the three samples of the corresponding inverse closest to the origin. Is the sample at the origin nonzero?

10. Let $X(z)$ be a rational function that does not have a pole at $z = \infty$. Show that $dX(z)/dz$ has a zero of order at least 2 at $z = \infty$. What can you say about the function $z\frac{dX(z)}{dz}$?

11. We are given the following information about an LTI system with frequency response $H(e^{j\omega})$: (a) the systems is causal, (b) $H(e^{j\omega}) = H^*(e^{-j\omega})$, (c) the DTFT of the sequence $h[n+1]$ is real. Using the information given can you determine whether or not the impulse response of the system is of finite-duration?

12. A causal LTI system is described by the difference equation $y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$, where a is real-valued and $|a| < 1$.
    (a) Find a value of b such that the frequency response of the system satisfies $|H(e^{j\omega})| = 1\ \forall\omega$. Such a system is called an *all-pass system*.
    (b) Roughly sketch $\angle H(e^{j\omega})$, $0 \le \omega \le \pi$ when $a = \frac{1}{2}$ and $-\frac{1}{2}$.
    (c) Find and plot the output of this system with $a = -\frac{1}{2}$ for the input $(1/2)^n u[n]$. This example shows that even though magnitude frequency response is constant over all frequency, i.e., no frequency component is changed in magnitude, the nonlinear phase can distort the signal significantly.

13. An LTI system with frequency response $H(e^{j\omega})$ has the following properties: (a) the systems is causal; (b) $H(e^{-j\omega}) = H^*(e^{j\omega})$; (c) the DTFT of $h[n+1]$ is real-valued; (d) $H(e^{j\pi}) = 0$; (e) $\int_{-\pi}^{\pi} H(e^{j\omega})\, d\omega = 4\pi$. From the above information, determine $h[n]\ \forall n$.

14. Consider

$$Y(z) = \frac{1-a^2}{1+a^2-a(z+z^{-1})}$$

Using long-division, find (i) causal inverse z-transform (quotient should be in powers of z^{-1}), and (ii) anti-causal inverse z-transform (quotient should be in powers of z). Compare these answers with those obtained by recursive solution of the corresponding difference equation assuming initial- and final-rest conditions.

15. An LTI system is described by the system function

$$H(z) = \frac{\left(1-\frac{1}{2}z^{-2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{2}$$

Determine the impulse response of the system and the difference equation that relates the input and output.

16. Determine the inverse z-transform using the specified method for each of the following: (a) $\dfrac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$; right-sided sequence, long-division, (b) $\dfrac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}}$; stable sequence, partial-fraction expansion, (c) $\ln(1-4z)$, $|z| < \frac{1}{4}$, power-series method.

17. The following facts are given about and LTI system characterized by $h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z)$: (i) $h[n]$ is real-valued, (ii) $h[n]$ is right-sided, (iii) $\lim_{z\to\infty} H(z) = 1$, (iv) $H(z)$ has two zeros, and (v) $H(z)$ has a complex-valued pole on the circle defined by $|z| = 3/4$. Is the system causal? Is it stable?

18. Let $x[n] = u[-n-1] + (1/2)^n u[n]$. It is convolved with $h[n]$, where $h[n] = 0$ for $n < 0$, and results in $y[n]$ whose z-transform is give by

$$Y(z) = \frac{-0.5z^{-1}}{(1-0.5z^{-1})(1-\alpha z^{-1})}$$

where $0.5 < |\alpha| < 1$. Determine $H(z)$ and its RoC. Also specify the RoC of $Y(z)$.

19. The input to a causal LTI system $H(z)$ is the signal $x[n] = -4\cdot 2^n u[-n-1] - 0.5^n u[n]$. The output signal's z-transform is given by

$$Y(z) = \frac{1-z^{-2}}{(1-0.5z^{-1})(1-2z^{-1})}$$

Determine (a) $H(z)$ and its RoC, and (b) the RoC of $Y(z)$.

20. The input to an LTI system is $x[n] = 2^n u[-n-1] + 0.5^n u[n]$, which produces the output $y[n] = 6\cdot(0.5^n - 0.75^n)u[n]$.

    (a) Determine the system function $H(z)$, its poles, zeros, and RoC.

(b) Write down the difference equation that relates $x[n]$ and $y[n]$.

21. Consider the filter $H(z)$ that corresponds to the system characterized by the difference equation $y[n] - y[n-2] = x[n] - x[n-6]$.
   (a) Determine the poles and zeros of $H(z)$. Is the filter causal? stable?
   (b) Derive and sketch the magnitude frequency response.

   Refer to the material on pp. 388–389 of S.K. Mitra's "Digital Signal Processing" (3rd edition, Tata McGraw-Hill, 2006).

22. Write down the difference equation (coefficients can be complex) whose homogeneous solution is $e^{j\omega_0 n}u[n]$. By considering the real and imaginary parts of the difference equation separately, identify two discrete-time systems (with real-valued coefficients) whose outputs are $\cos\omega_0 n\; u[n]$ and $\sin\omega_0 n\; u[n]$. Comment on the order of the various difference equations.

23. **Computer assignment** Plot the real/imaginary parts, as well as magnitude/phase for various values of r and θ of the following DTFT:

$$G(e^{j\omega}) = \frac{1}{1 - 2r(\cos\theta)\, e^{-j\omega} + r^2\, e^{-j2\omega}}$$

   $\theta, \omega \in [0, 2\pi)$. In particular, observe what happens when (a) r is close to unity versus well away, and (b) θ is close to 0 (or π) versus $\pi/2$ (or $3\pi/2$). If you consider the denominator as a polynomial in $e^{j\omega}$, where are its roots? Can you relate the peak locations of the DTFT and the root locations? Although ω is a continuous variable, on a computer you will have to discretize it, which is typically done by taking N uniformly spaced points in $[-\pi, \pi)$.

24. **Computer assignment** Study the MATLAB function `freqz`. Consider the $G(e^{j\omega})$ given in the previous problem. Let `G = freqz(1, [1, -2*r*cos(theta), r*r], 4000)`. Plot the magnitude and phase of `G` and compare with the result of the previous experiment. Also plot `20*log10(abs(G))` and compare it with it linear-scale counterpart.

25. **Computer assignment** Consider the command `roots(poly(a))`, where `a` is a vector containing roots. Consider a root at 0.9 with multiplicity ranging from 1 to 6, and observe the accuracy of the answer (use `format double`) to observe more digits than the default display precision. "`roots(poly(1:20))` generates Wilkinson's famous example" (see also `http://www.ima.umn.edu/~arnold/455.f97/labs/lab02.ps`).

26. **Computer assignment** Carry out the partial fraction expansion of the transforms in P4.9 (p. 113 of the book "DSP Using MATLAB" by Proakis and Ingle, Brooks/Cole, 2000). Use the `residue` command.

27. **Computer assignment** Using MATLAB try problem M 6.1 on p. 351 of S.K. Mitra's "Digital Signal Processing" (3rd edition, Tata McGraw-Hill, 2006). Also lookup the `zplane` command.