<u>EC5142: Introduction to DSP</u>

<u>Problem Set 5</u>

(October 5, 2011)

1. An *analytic signal* has a one-sided spectrum. Let $z(t) = x(t) + j\,\hat{x}(t)$; $x(t)$ is real-valued and $\hat{x}(t)$ represents the *Hilbert transform* of $x(t)$. Show that $Z(\Omega)$ is zero for $\Omega < 0$. The *Hilbert Transform* of $x(t)$ is given by $\hat{x}(t) = x(t) * \dfrac{1}{\pi t}$.

   Hint: What is the Fourier transform of $\dfrac{1}{\pi t}$? Next, to find the Fourier transform of $\hat{x}(t)$, you can use the property that if two signals are convolved in the time domain, their transforms multiply.

2. Consider the even symmetric trapezoidal function $x(t)$ whose definition for $t \geq 0$ is: $x(t) = 1$, $0 \leq t \leq 1$, and linearly decaying to zero in the range $1 < t \leq 2$. $x(t) = 0$ for $|t| \geq 2$. What is $x''(t)$? Find $X(\Omega)$ by using the differentiation property.

   Note that $X(\Omega)$ decays as $1/|\,\Omega|^2$. In general, if the n-th derivative of $x(t)$ contains impulses, then the asymptotic behaviour of $X(\Omega)$ will be $O(|\,\Omega|^{-n})$. Recall how the spectrum of a rectangular pulse decays asymptotically.

   $x(t)$ can also be represented as a difference of two triangular pulses. A triangular pulse is the result of convolving a rectangular pulse with itself, which suggests an easy way to compute its transform.

3. Consider the voltage signal $v(t) = e^{-a|t|}$ $(a > 0)$. Find $V(\Omega)$. Calculate the energy of $v(t)$. If $v(t)$ were applied to an 1Ω resistor through an ideal LPF with cutoff frequency $\Omega_c = a$, find the energy of the output signal.

4. The following brief derivation shows that the area under the derivative is zero: $\displaystyle\int_{-\infty}^{\infty} x'(t)\,dt = j\Omega X(\Omega)|_{\Omega=0} = 0$. Confirm that this is indeed so, or find the fallacy in the reasoning.

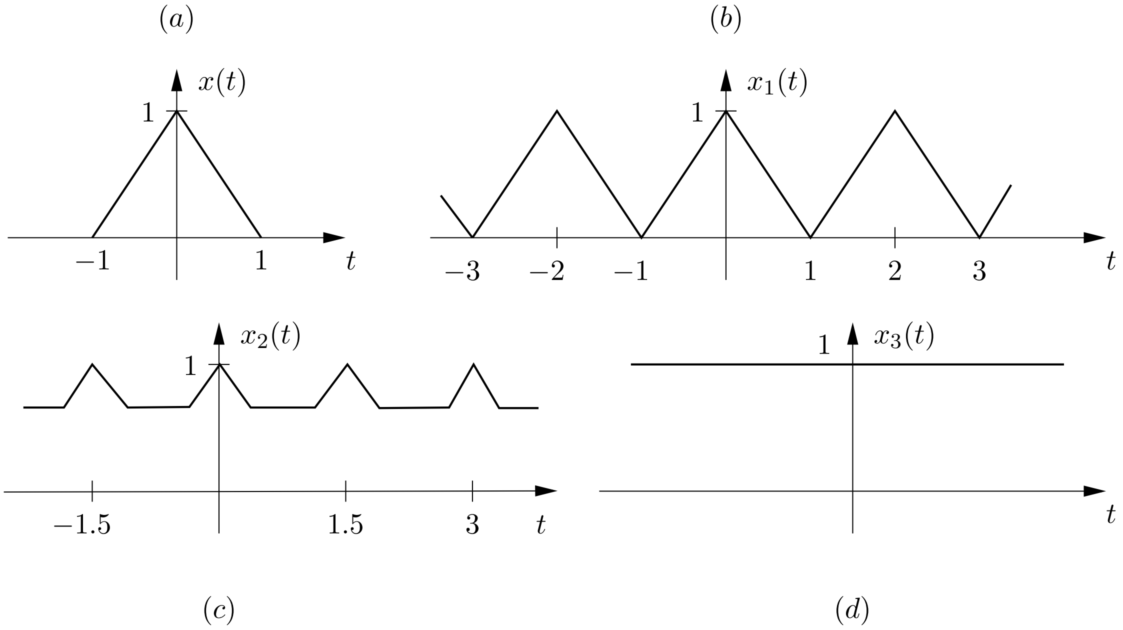
5. Find $f(t)$ when $F(\Omega)$ equals (a) $\displaystyle\sum_n \delta(\Omega - n\Omega_0)$, and (b) $\dfrac{10}{(j\Omega+1)(j\Omega+2)}$.

6. Show that the Fourier transform of the Gaussian signal $x(t) = e^{-t^2}$ is self-similar, i.e., also Gaussian. Ponder about other self-similar transforms.

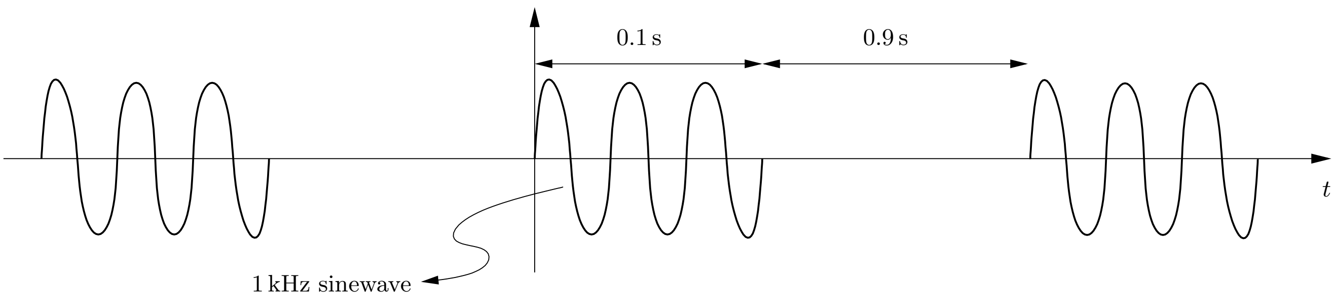
7. Using the time-convolution theorem find the inverse Fourier transform of $X(\Omega) = \dfrac{1}{(a+j\Omega)^2}$. Can this also be solved using the differentiation property?

8. Consider an ideal lowpass filter with frequency response $H(\Omega) = 1$ for $|\Omega| < 4\pi$ and zero otherwise. To this filter we apply a periodic square wave whose fundamental period is defined as $x(t) = 10$ for $0 < t < 1$ and $x(t) = 0$ for $1 < t < 2$. Find the output $y(t)$.

9. The Fourier transform of the triangular pulse shown in Fig. (a) below is $X(\Omega) = \left(\frac{\sin\frac{\Omega}{2}}{\frac{\Omega}{2}}\right)^2$. Using this information, find the Fourier series coefficients of the periodic waveforms shown in Figures (b), (c), and (d).



10. A unit amplitude sinewave of 1 kHz frequency is multiplied by a periodic "on–off" square wave that stays "on" for 0.1 s and "off" for 0.9 s (see figure below, which is not to scale). Find the frequency components present in the signal. Find the frequency component with the largest amplitude; what is the value of this amplitude? Hint: Use the modulation property.



11. Suppose we are given the following facts about a signal $x(t)$:
   (a) $x(t)$ is real-valued.
   (b) $x(t+6) = x(t)$ and has Fourier coefficients a_k.

(c) $a_k = 0$ for $k = 0$ and $k > 2$.

(d) $x(t) = -x(t-3)$.

(e) $\frac{1}{6}\int_{-3}^{3} |x(t)|^2\, dt = \frac{1}{2}$

(f) a_1 is a positive real number

Show that $x(t) = A\cos(Bt + C)$, and determine the values of A, B, and C.

12. The system function $H(\Omega) = 2 + \delta(\Omega) + \frac{1}{j\pi\Omega}$. Find the response of the system to (a) $\delta(t)$, and (b) $te^{-t}u(t)$.

13. Find the Fourier transform of the following signals: (a) $(1 - e^{-t})u(t)$, (b) $\cos(\pi t)/(\pi t)$, (c) $\mathrm{sech}(\pi t)$, (d) $1/\sqrt{|t|}$. You may have to refer to books like, "Tables of Integrals, Series, and Products" by Gradshteyn and Ryszhik for evaluating some of these integrals.

14. Let $x(t) = 2F_c\,\mathrm{sinc}(F_c t)$. Evaluate (a) $x(t) * x(t)$, and (b) $\int_{-\infty}^{\infty} x(t)\, dt$.