<u>EC5142: Introduction to DSP</u>

<u>Problem Set 3</u>

(September 1, 2011)

1. (a) Suppose the complex-valued periodic signal possesses conjugate symmetry, i.e, $x(t) = x^*(-t)$. What property do the FS coefficients possess?

   (b) Now suppose we define "evenness" for a *complex-valued* signal as $x(t) = x(-t)$. Do the FS coefficients of $x(t)$ possess any special property?

   (c) For a complex-valued $x(t)$ can you now see why we defined "(conjugate) even" as $x(t) = x^*(-t)$ instead of $x(t) = x(-t)$?

2. Even though $\cos \Omega_0 t$ and $\sin \Omega_0 t$ have the "same frequency content", their FS coefficients are different. The coefficient magnitudes are same but their phases are different. What frequency components are present in $\sin^2 \Omega_0 t$ and $\cos^2 \Omega_0 t$? Once again carefully compare their FS coefficients.

3. The trigonometric form of the Fourier series for the real-valued periodic function $x(t)$ is given by $a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)]$. Show that

$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_{T_0} x(t)\, dt \\ a_n &= \frac{2}{T_0} \int_{T_0} x(t) \cos(n\Omega_0 t)\, dt \\ b_n &= \frac{2}{T_0} \int_{T_0} x(t) \sin(n\Omega_0 t)\, dt \end{aligned}$$

4. If $x(t) = -x(t + T/2)$, what can you about the Fourier series coefficients a_k for even k ? Justify your answer.

5. Let $x(t)$ be a real-valued periodic signal with period T. Find the FS coefficients of the following signals in both exponential form and trigonometric form.

   (a) $x_1(t) = A$ for $0 < t < T/2$ and zero for $T/2 < t < T$.

   (b) $x_2(t) = A$ for $|t| < T/4$ and zero otherwise (over the fundamental period). What is the relationship between $x_1(t)$ and $x_2(t)$? What is the relationship between their FS coefficients?

   (c) Let $x_3(t) = -A/2$ for $-T/2 < t < 0$ and equal to $A/2$ for $0 < t < T/2$. What is the relationship between $x_1(t)$ and $x_3(t)$? What is the relationship between their FS coefficients?

   (d) $x_4(t) = A$ for $|t| < d$ (where $d < T/2$) and zeros otherwise (over the fundamental period). How do the FS coefficients change as d varies over the interval $[0, T/2]$?

(e) $x_5(t)$ is similar to $x_1(t)$ except that it is defined to be A for $0 \le t \le T/2$ and zero for $T/2 < t < T$. Do the FS coefficients change? Ponder about the equality sign in the expression $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$.

6. Signal symmetry imposes some properties on the Fourier series coefficients. Verify the properties given in the table below, in which entries such as $a_0 \neq 0$ and $b_{2n+1} \neq 0$ are to be interpreted to mean that these coefficients are not necessarily 0 but may be so in specific examples.

Table 1: **Effects of Symmetry**

Symmetry	a_0	a_n	b_n	**Remarks**
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ only, and multiply the coefficients by 2
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ only, and multiply the coefficients by 2
Half-wave odd	$a_0 = 0$	$a_{2n} = 0$, $a_{2n+1} \neq 0$	$b_{2n} = 0$, $b_{2n+1} \neq 0$	Integrate over $T/2$ only, and multiply the coefficients by 2

7. Let $x_1(t+T) = x_1(t)$ and $x_2(t+T) = x_2(t)$. The Fourier series coefficients of $x_1(t)$ are given by c_k, whereas those of $x_2(t)$ are given by d_k. Express $\dfrac{1}{T}\int_T x_1(t) \cdot x_2(t)\, dt$ in terms of c_k and d_k.

8. Find the Fourier series representation of (a) $|\sin 2\pi t|$ (full-wave rectified sine), and (b) $\sin 2\pi t$ for $0 \le t \le 0.5$ and 0 for $0.5 \le t \le 1$ (half-wave rectified sine).

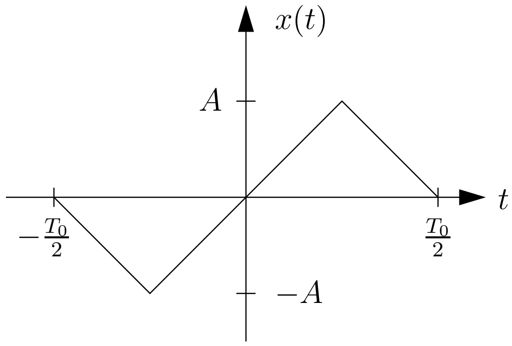
9. $x(t)$ is periodic with period T and linearly increases from $x(0) = 0$ to $x(T) = E$.
   (a) Find its Fourier series expansion in both exponential and trigonometric forms.
   (b) Find the FS coefficients of $dx(t)/dt$, making sure to take care of any impulses that may be present. Knowing the coefficients of $dx(t)/dt$, can you find those of $x(t)$?

10. $y(t)$ is such that $y(t+T) = y(t)$, and linearly falls from $y(-T/2) = 3E$ to $y(T/2) = E$. Find its FS coefficients without explicitly computing them; instead, derive them using the result from the previous problem and using properties of Fourier series.

11. A series combination of $1\,\Omega$ and $1\,\mathrm{F}$ is excited by a periodic square wave with period $1\,\mathrm{s}$. At $t = 0.5$ the input voltage switches from E_0 to $-E_0$. Find the FS expansion of the loop current $i(t)$ under steady-state conditions. Express the power dissipated in the circuit as a sum of harmonic components.

12. Let $x(t) = \sin(2\pi f_0 t)$ for $0 \le t \le 0.1$ seconds and zero for $0.1 < t \le 0.9$ seconds; $x(t+1) = x(t)$. Find that frequency component with the largest spectral magnitude.

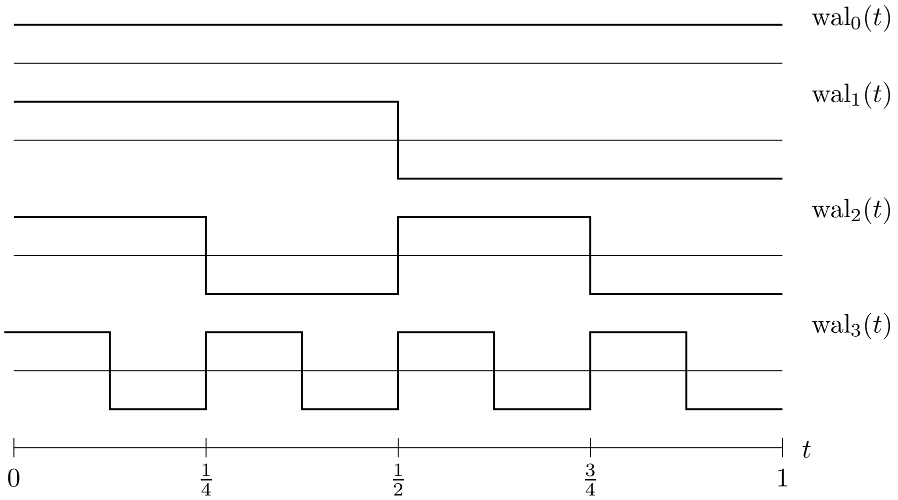
13. One period of a triangular wave is given in the figure below. Compute its Fourier series coefficients in both exponential and trigonometric forms. How do the a_k fall off with increasing k? Compare its rate of fall with that of a periodic square wave.



14. (a) Let $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Plot $x(t)$. Find its FS coefficients in both exponential and trigonometric forms.

    (b) Let $x(t) = 1$ for $t = 0, \pm T, \pm 2T, \ldots$ and zero otherwise. Find the FS coefficients in both exponential and trigonometric forms.

15. Walsh functions are a set of orthogonal functions defined over the interval $[0, 1)$ that take on the values of ± 1. These functions are characterized by their *sequency*, which is defined as one-half the number of zero-crossings of the function over the interval $[0, 1)$. The first few Walsh basis functions are shown in the figure below.



    (a) Verify that the Walsh functions are orthogonal over $[0, 1)$.

(b) Suppose we wish to represent $x(t) = t\,[u(t) - u(t-1)]$ in terms of the following Walsh function expansion:

$$x(t) = \sum_k c_k \,\mathrm{wal}_k(t)$$

Find the coefficients c_k for $k = 0, 1, \ldots, 6$. Repeat for (a) $x(t) = \sin 2\pi t$, and (b) $x(t) = \mathrm{wal}_3(t)$.

16. **Computer Experiment** The Fourier series representation of a square wave with half-wave odd symmetry is given by $\sum_{k=1,3,5,\ldots} (1/k)\,\sin(kt)$. The aim of this exercise is to plot the partial sums and see how the series converges at a discontinuity. Define `t=(0:1e-4:2*pi)';` (use `%pi` in Scilab). The following lines of code compute the partial sums: `N=5; x=zeros(62832,1); for k=1:N, x = x + 1/(2*k-1)*sin((2*k-1)*t); end`. Plot the results for various values of `N` and observer the behaviour. In particular, note the height of the overshoot.

17. **Computer Experiment** For an RC lowpass filter $H(\Omega) = \dfrac{1}{1 + j\Omega RC}$. To this we feed an odd-symmetric unit amplitude square wave with period $T_0 = 1$, having Fourier series coefficients a_k (which you should compute analytically). The output $y(t) = \sum_k a_k H(k\Omega_0) \exp(jk\Omega_0 t)$. Make the number of terms in the sum to be a variable L. Let $RC = 0.2$ and 0.05. Plot one period of the output for different values of L. If the value of RC is very large (e.g., 2), what is the shape of the output? What operation has the circuit performed on the input in this case? Observe how the shape depends on the number of terms in the summation. In particular, what is the dependence of the sharpness of the corners on the number of terms?

    Compare these results by convolving numerically a 10-second square wave train with $h(t) = 1/(RC) \exp(-t/RC)\, u(t)$. A 10-second square wave can be easily constructed by first defining `t=(-0.5:0.001:0.499)';` and getting one period by `x1=sign(t);` (`sign` is a Matlab/Scilab function). Next compute `tmp=x1*ones(1,10)`. The ten columns of `tmp` are identical and equal to `x1`. Form the periodic signal `xp` by `xp=tmp(:);` which stacks all the columns into a single vector. Verify the result by `plot((-5:0.001:4.999)',xp)`. In the convolution output observe the pulse shapes at the beginning, middle, and ending portions.

    Repeat the experiment for an RC highpass filter (output is taken across the resistance).

    Repeat all of the above when the input is a triangular wave.