EC5142: Introduction to DSP

Problem Set 2

(August 25, 2011)

1. (a) Simplify the expression for $x(t)\delta(t)$ when $x(t)$ equals: (i) t, and (ii) $\sin t$. Hint: For the first case, consider the limit of $t\,\delta_\Delta(t)$ as $\Delta \longrightarrow 0$, where $\delta_\Delta(t)$ is a symmetric pulse of height $1/\Delta$ and width Δ centred at the origin, i.e., $\delta_\Delta(t) = \frac{1}{\Delta}\left[u\left(t+\frac{\Delta}{2}\right) - u\left(t-\frac{\Delta}{2}\right)\right]$.

   (b) What is the value of $\delta(t-\pi)\cos t$?

2. The *signum* function (or the "sign function") $\operatorname{sgn}(t)$ is defined as being equal to 1 for $t > 0$ and equal to -1 for $t < 0$. Find the derivative of $\operatorname{sgn}(t)$. What is the derivative of $\operatorname{sgn}(-t)$?

3. The impulse itself can be differentiated. To get a feel for this, consider the symmetric triangle centred at the origin with width 2Δ and height $1/\Delta$.

   (a) What is the limit of the above triangular pulse as $\Delta \longrightarrow 0$?

   (b) Differentiate the triangular pulse and then take the limit as $\Delta \longrightarrow 0$. Draw a plot of the limit, inspired by the picture of $\delta(t)$. What is the area under $\delta'(t)$?

4. Simplify, if possible, the following convolutions: (a) $u(t) * u(t)$, and (b) $u(t) * u(-t)$.

5. Let $s(t) = p(t) * [q(t) * r(t)]$. Express $s(t)$ as a double integral given that $q(t) * r(t) = \int_{-\infty}^{\infty} q(\lambda)\, r(t-\lambda)\, d\lambda$

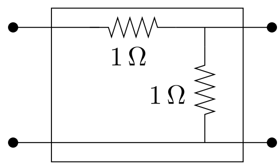
6. Associativity of convolution holds only under certain conditions. To see that convolution is not always associative, let $p(t) = u(t)$, $q(t) = -t\, e^{-t^2/2}$, and $r(t) = u(-t)$. Show that the following are well-defined: (i) $p(t) * q(t)$, (ii) $[p(t) * q(t)] * r(t)$, (iii) $q(t) * r(t)$, and (iv) $p(t) * [q(t) * r(t)]$. Nevertheless, $[p(t) * q(t)] * r(t) \neq p(t) * [q(t) * r(t)]$.

7. Let $p[n] = \alpha^n$, $q[n] = \delta[n] - \alpha\,\delta[n-1]$, $r[n] = \alpha^n\, u[n]$, where $\alpha \neq 0$ is a complex number. Show that the following are well-defined: (i) $p[n] * q[n]$, (ii) $(p[n] * q[n]) * r[n]$, (iii) $q[n]*r[n]$, and (iv) $p[n]*(q[n] * r[n])$. Nevertheless, $(p[n] * q[n])*r[n] \neq p[n]*(q[n] * r[n])$.

8. (a) Suppose we have two LTI systems connected in *cascade*, with the impulse response of the individual systems being $h_1(t)$ and $h_2(t)$. Show that the overall impulse response is $h_1(t) * h_2(t)$.

   (b) Suppose that we now connect the two LTI systems in *parallel*. Show that the overall impulse response is $h_1(t) + h_2(t)$.

(c) Note that when two LTI systems are in cascade, the overall impulse response being $h_1(t) * h_2(t)$ assumes that the second system does not alter the first system when it is connected. To see this, consider the "ladder section" shown on the right. What is $h(t)$, the impulse response of a single ladder section? Now suppose that two such sections are connected in cascade. What is the expected overall impulse response? Now using loop- or mesh-analysis, find the actual impulse response. Is it equal to $h(t) * h(t)$? If you want the cascaded system's impulse response to be $h(t) * h(t)$, how should the two sections be connected?



Ladder Section

9. The *cross-correlation* of two real-valued signals $x(t)$ and $w(t)$ is defined as

$$\phi_{xw}(t) = \int_{-\infty}^{\infty} x(t+\tau)w(\tau)\, d\tau$$

The special case when $x(t) = w(t)$ is called the *autocorrelation* function.

(a) Express $\phi_{xw}(t)$ as a convolution.

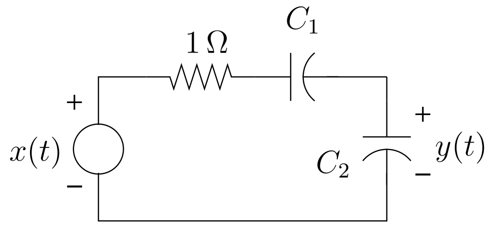
(b) Find the relationship between $\phi_{xw}(t)$ and $\phi_{wx}(t)$ and also between $\phi_{xx}(t)$ and $\phi_{xx}(-t)$.

(c) Let $x(t) = u(t) - u(t-1)$. Plot (a) $y(t) = x(t) * x(t)$, and (b) $\phi_{xx}(t)$.

10. Consider a series RLC circuit with values $R = 1\,\Omega$, $C = 1/2\,F$, and $L = 1\,H$. The input voltage is $x(t)$ and the loop current is $y(t)$. Write the input-output differential equation for this circuit. Now suppose that $x(t) = 10\,e^{-3t}$, which is applied at $t = 0$. Further, $y(0^-) = 0$ and $v_C(0^-) = 5$. Find $y(t)$ for $t \geq 0$.

11. Consider the circuit shown on the right.

(a) Determine the differential equation relating the input $x(t)$ and the output $y(t)$.



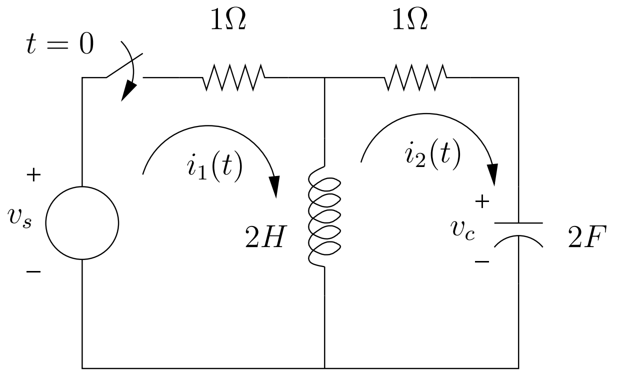
(b) Determine the closed-form expression for $y(t)$ in response to $x(t) = 4t\,e^{-3t/2}u(t)$. Assume $C_1 = 1\,F$ and $C_2 = 2\,F$; $v_{C_1}(0^-) = 2\,V$ and $v_{C_2}(0^-) = 1\,V$.

12. Consider the circuit shown on the right.

(a) The switch is closed at $t = 0$. Write the differential equations for the two loops.

(b) Taking $i_2(t)$ as the output and $v_s(t)$ as the input, write the differential equation relating the two.

(c) If $v_s(t) = 33\cos(t)$, find $i_2(0^+)$ and $i_2'(0^+)$, assuming zero inductor current at $t = 0^-$ and $v_c(0^-) = 5V$.



13. (a) Working in the time-domain and using a step-by-step procedure, assuming $x[n] = \delta[n]$, find the output the system described by the following difference equation: $y[n] - 0.8\,y[n-1] = x[n]$. Assume $y[-1] = 0$.

    (b) Repeat when the input is $0.5\,u[n]$; as before, assume $y[n] = 0$ for $n < 0$. Compare this answer with the result of $0.5\,u[n] * 0.8^n\,u[n]$? Can you relate this to the system output obtained in part (a) ?

14. **Computer Experiment** To generate a sequence with frequency components 0.1 and 0.2, first define `n=(0:99)';` to get a column vector of length 100. The sequence `x` can then be generated using the command `x = cos(2*pi*0.1*n+pi/3) + sin(2*pi*0.2*n + pi/8);`. The higher frequency component can be eliminated by convolving `x` with `h`, where `h = 0.2*ones(5,1);`. Plot `y` where `y = conv(x,h)`. Verify that the second component is practically absent in `y`. Give a theoretical explanation as to why the convolution with an all ones sequence of length 5 has eliminated the sinusoid with frequency 0.2.

15. **Computer Experiment** Form the sequence $h[n] = a^n\,u[n]$ for $a = 0.9$. Let $x[n] = \sin(2\pi n/16)$ for $n = 0, 1, ..., 999$. Compute the convolution of these two sequences and plot the first hundred samples. The built-in commands for convolution are `convol` (Scilab) and `conv` (Matlab).

    Calculate the theoretical convolution output when $h[n] = (0.9)^n\,u[n]$ is convolved with $x[n] = \sin(2\pi n/16)\ \forall n$. The result can be decomposed into two terms: $\alpha\,0.9^n\,u[n] + \beta\,\sin(2\pi n/16 + \gamma)$. Calculate the constants α, β, and γ and compare the theoretical results with that obtained from Matlab/Scilab.