Abstract—We consider the problem of jointly recovering block sparse signals that share the same support set, using multiple measurement vectors (MMV). We consider the generalized MMV (GMMV) model, where the different measurement vectors could have been obtained using different sensing matrices. We study greedy and convex programming based recovery algorithms and theoretically establish their support recovery guarantees. Our results present insights on how the correlation between the block sparse signals plays a role in the recovery performance. Next, we consider the problem of cell search in heterogeneous cellular networks (HetNets). With the cell search process, the mobile terminal (MT) acquires the synchronization parameters such as frame timing, residual frequency offset and the physical layer identity of a base station (BS) suitable for its connection. In HetNets, due to the increased density of BS, the MT may receive strong interference from several BS in its neighborhood. We establish that the problem of cell search in HetNets can be solved using the GMMV joint block sparse signal recovery framework. We numerically study the performance of the cell search algorithms proposed using our framework and show that they perform significantly better than the successive interference cancellation algorithm existing in the literature.

Keywords—block sparse signal recovery, multiple measurement vectors, generalized multiple measurement vectors, greedy algorithms, downlink synchronization, heterogeneous cellular networks

I. INTRODUCTION

Compressive sensing (CS) deals with the problem of recovering a sparse unknown vector \( x \) from an under-determined system of linear measurements, possibly corrupted by noise \( y = Ax + w \). CS theory [1], [2] establishes the conditions on the sensing matrix \( A \) under which, the recovery of sparse vector \( x \) (up to an accuracy depending on the noise power) is possible using practical (polynomial-complexity) algorithms. The restricted isometry property (RIP) of the sensing matrices plays a crucial role in establishing the theoretical guarantees of several recovery algorithms including basis pursuit (\( \ell_1 \) norm minimization) [3] and matching pursuit (MP) [4], [5]. Another relevant metric is the mutual incoherence of the sensing matrix [6], [7] which is relatively simple and easy to characterize compared to the RIP property. Matching pursuit and orthogonal matching pursuit (OMP) [4], [8] are some of the algorithms whose recovery guarantees are established in terms of mutual incoherence.

There are many applications where the non zero entries in the unknown vectors appear in groups or clusters [9]. Sparse signals exhibiting such a structure are called block sparse signals. Exploiting this additional structure in the unknown signals, several block sparse signal recovery algorithms have been developed in the literature [10]–[13]. Further, the block RIP and mutual subspace incoherence properties of the sensing matrices have been shown to play crucial roles in the recovery guarantees.

Another line of work addresses sparse signal recovery from multiple measurement vectors (MMV) [14], which deals with recovering multiple sparse signals that share the same sparsity pattern, i.e., the same support set for the non-zero entries. A different approach for the MMV problem is proposed in [15], which splits the MMV into a series of single measurement vector (SMV) problems. The work done in [16] extends the recovery from MMV to noisy measurements and also considers the correlation among the observations. In [17], a generalized MMV (GMMV) model is considered where the sensing matrices differ for different measurement vectors. In these previous works [14], [17], [18] the recovery algorithms for SMV models are extended to the MMV/GMMV models and their corresponding recovery guarantees are provided.

In our work, we consider the problem of jointly recovering a set of block sparse signals that share the same sparsity pattern, for the general case of GMMV models. We extend the previously developed block sparse algorithms - subspace correlation pursuit (SCP) [19], subspace matching pursuit (SMP) [13] and single step SMP [19], for the GMMV models. We also consider a convex-program based relaxed norm minimization algorithm to the joint block sparse signal recovery problem. For all these algorithms, we establish sufficient conditions for perfectly recovering the support of the non-zero blocks.

In the second part of the paper, we consider an application of the joint block sparse recovery problem. Specifically, we establish that the initial cell search problem (also referred as downlink synchronization) in heterogeneous cellular networks (HetNets) [20], [21] can be solved using joint block sparse signal recovery framework. In HetNets, in addition to the conventional macro base stations (BS) multiple low power BS like pico, femto and relay nodes with smaller coverage area are deployed. Due to the increased density of base stations in the neighborhood of a mobile terminal (MT), and due to the differences in power levels of the various BS, interference...
becomes a key issue in HetNets.

Every BS bears a unique cell identity and periodically sends synchronization signals (SS) specific to its identity in every frame. Before the mobile terminal (MT) can connect to its suitable BS, MT must complete the cell search or downlink synchronization process, by which MT recovers the following synchronization parameters, 1) location of start of a frame - timing detection, 2) cell identity of the desired BS 3) estimation of the frequency offset between the desired BS and MT oscillators. The interference in HetNets, caused due to the increased density of base stations, poses severe challenges for the cell search process. We specifically consider the 3GPP LTE cellular network frame structure. With single receive antenna, using single frame of observation, the cell search problem was solved in [19] using SVM block sparse signal recovery formulation. In this paper, when the MT has multiple receive antennas and/or multiple frames of observations, we solve the LTE cell search problem using GMMV block sparse recovery framework. The proposed cell search algorithms based on the joint block sparse recovery show significant gains over conventional cell search algorithms.

Notation: matrices/vectors are denoted by bold upper-case/lowercase letters, \( \| \cdot \| \) for \( \ell_2 \) norm, \( \ell_p \) norm by \( \| \cdot \|_p \), Frobenius norm by \( \| \cdot \|_F \), transpose by \( \cdot^t \), hermitian by \( \cdot^* \), column space of matrix \( A \) by \( \mathcal{R}(A) \), sets by mathcal font \( A \), cardinality of the set by \( |A| \), and set minus operation by \( A \setminus B \).

II. System Model and Problem Statement

Let \( x \) be a vector of size \( CL \times 1 \), which is constructed by vertical concatenation of \( C \) blocks \( \{ x_i, i = 1, \cdots, C \} \), with each block of size \( L \times 1 \). Let the set \( S \) denotes the set of indices corresponding to the non-zero blocks in \( x \), i.e., \( S = \{ i \mid \| x_i \| > 0 \} \). We call the vector \( x \) as \( K \)-block sparse vector if there are only at-most \( K \) non-zero blocks in \( x \), i.e., \( |S| \leq K \) and we refer the support set \( S \) as the sparsity pattern of \( x \). In this paper, we are interested in block sparse vectors which share the same sparsity pattern. Towards that, let \( \{ x^{(m)}, m = 1, \cdots, M \} \) be \( K \)-block sparse vectors of size \( CL \times 1 \), each composed of \( C \) blocks \( \{ x_i^{(m)} \} \) of size \( L \times 1 \) with their sparsity patterns \( S^{(m)} \) being identical, say, \( S^{(1)} = \cdots = S^{(M)} = S \). Similarly, let the sensing matrix \( A^{(m)} \) of size \( N \times CL \) be constructed as horizontal concatenation of \( C \) matrices (blocks) \( A_i^{(m)} \) each of size \( N \times L \), such that \( A^{(m)} = \begin{bmatrix} A_1^{(m)} & \cdots & A_C^{(m)} \end{bmatrix} \). We consider the following noisy observation model,

\[
y^{(m)} = A^{(m)} x^{(m)} + w^{(m)}, \quad m = 1, \cdots, M. \quad (1)
\]

The above model is called multiple measurement vector (MMV) model if all the sensing matrices are identical, i.e., \( A^{(1)} = \cdots = A^{(M)} = A \) (say). On the other hand, if all the measurement matrices are not identical, \( A^{(i)} \neq A^{(k)} \) for some \( i, k \in \{ 1, \cdots, M \} \), then we refer the model as generalized multiple measurement vector (GMMV) model. With some loss of generality, we assume that the columns of each block \( A_i^{(m)} \) are linearly independent and have unit norm. We also assume that norm of the additive noise is bounded as \( \| w^{(m)} \| \leq \lambda \).

The objective of our work is to develop algorithms to recover the sparsity pattern of the vectors \( \{ x^{(m)} \} \) (utilizing their joint sparse structure) using the observations \( \{ y^{(m)} \} \) based on the model given in (1).

The several special cases of this problem have already been studied in the literature. For instance, with \( L = 1, M = 1 \), we get the conventional sparse signal recovery problem [1], [2]. With \( M = 1, L > 1 \), we get the block sparse signal recovery problem using single measurement vector studied in [11], [12]. For conventional sparse signals (\( L = 1 \)), the multiple measurements with same sensing matrix (MMV) case was analyzed in [14], [15], [22] and the different sensing matrices (GMMV) case was addressed in [17].

In this paper, we study the GMMV block sparse signal recovery problem. In Section III, we present greedy and convex programming based recovery algorithms and establish the guarantees on their recovery performance. As a special case, our results apply to the MMV model as well. In Section IV, we establish that the task of cell search in LTE HetNets with multiple antennas and/or observation frames can be solved using GMMV block sparse signal recovery framework. In Section V, we compare the LTE cell search performance using block sparse recovery algorithms and the conventional successive interference cancellation algorithm.

III. Recovery Algorithms

In this section, we propose greedy and convex programming based recovery algorithms for the model (1) and establish their recovery guarantees. It can be seen that each measurement \( y^{(m)} \) is a superposition of (at-most) \( K \) vectors with each vector belonging to one of the column spaces \( \{ \mathcal{R}(A_i^{(m)}) \} \). The active blocks from \( \{ A_i^{(m)} \} \) used in the construction of \( y^{(m)} \) depends on the non-zero blocks from \( \{ x_i^{(m)} \} \). Since all the block sparse vectors \( \{ x^{(m)} \} \) share the same sparsity pattern, the location/indices of the active blocks is the same in all the sensing matrices \( \{ A^{(m)} \} \). Our recovery algorithms jointly recover the support of the non-zero blocks in \( \{ x^{(m)} \} \) using all the measurement vectors \( \{ y^{(m)} \} \).

A. Subspace Correlation Pursuit

Subspace correlation pursuit (SCP) is a greedy block sparse signal recovery algorithm [19] and is also referred as block OMP algorithm [12]. We extend SCP algorithm for our GMMV model and obtain SCP-GMMV algorithm. SCP-GMMV correlates the observations \( y^{(m)} \) with each sub-block \( A_i^{(m)} \) and choose the one which maximizes sum of the correlations over all \( m = 1, \cdots, M \). The component of the detected subspace is subtracted out from each observation and the algorithm proceeds iteratively as follows.

Step 1. Initialize \( k = 1 \), \( \hat{S} = \emptyset \), \( \Phi_0 = [ \cdot ] \), \( r_0^{(m)} = y^{(m)} \).

Step 2. Find the index \( q \) such that

\[
q_k = \arg \max_{i=1,\cdots,C} \sum_{m=1}^{M} \| A_i^{(m)} r_k^{(m)} \|.
\]

Step 3. \( \hat{S} \leftarrow \hat{S} \cup q_k \).

Step 4. Concatenate: \( \Phi_k^{(m)} = [ \Phi_k^{(m-1)} A_i^{(m)} ] \).

Step 5. \( \hat{x}^{(m)} = \arg \min_{x^{(m)}} \| y^{(m)} - \Phi_k^{(m)} x^{(m)} \| \).
Step 6. \( r_k^{(m)} \leftarrow y^{(m)} - \Phi_k^{(m)} x^{(m)} \) and \( k \leftarrow k + 1 \).

Step 7. If \( k \geq K \) or \( \sum_{m=1}^{M} \| r_k^{(m)} \| \geq \tau \), go to Step 2 else stop.

When the algorithm stops, \( \hat{S} \) gives the recovered sparsity pattern and reconstructed block sparse vector \( \hat{x}^{(m)} \) can be obtained by suitably adding zero blocks corresponding to the indices \( i \notin \hat{S} \) with \( \hat{x}^{(m)} \) which gives the \( m \)-th reconstructed vector. \( \hat{x}^{(m)} \) is essentially the least squares estimate of \( x^{(m)} \) from \( y^{(m)} \) given that the sparsity pattern is \( \hat{S} \). If the exact number of \( K \) is not known as in typical applications, the stopping criterion can solely depend on the norm of the residual. The parameter mutual incoherence of the sensing matrices \( A^{(m)} \) play crucial role on the recovery guarantee of SCP-GMMV.

**Definition 1.** The mutual incoherence (\( \gamma \)) of a matrix \( A^{N \times CL} \) is defined as,

\[
\gamma = \max_{i, k \in \{1, \ldots, CL\}} \left\{ \frac{\| r_i \| \| r_k \|}{\| r_i \| + \| r_k \|} \right\}
\]

where \( r_i \) denotes the \( i \)-th column of \( A \).

\( \gamma \) captures the smallest angle between the columns of the matrix. By denoting the mutual incoherence of the sensing matrix \( A^{(m)} \) as \( \gamma^{(m)} \), the recovery guarantee of SCP-GMMV is given as follows.

**Theorem 1.** SCP-GMMV perfectly recovers the sparsity pattern of \( K \)-block sparse vectors from the observation model in (1), if \( K < \)

\[
\min_{k \in \hat{S}} \frac{1}{2L} \left( 1 + \frac{1}{\gamma} \right) - \frac{M \lambda}{\sqrt{L} \sum_{m=1}^{M} \gamma^{(m)}} \| x_k^{(m)} \| \]

**Proof:** See Appendix A.

As a special case, for the MMV model \( A^{(m)} = A \forall m \), the above algorithm (which we refer as SCP-MMV) has the following recovery guarantee.

**Corollary 1.** Using identical sensing matrices \( A^{(m)} = A \forall m \) with mutual incoherence \( \gamma \), SCP-MMV perfectly recovers the sparsity pattern of the \( K \)-block sparse vectors if,

\[
K < \frac{1}{2L} \left( 1 + \frac{1}{\gamma} \right) - \frac{\lambda}{\sqrt{L} \gamma b_{min}}
\]

where \( b_{min} = \min_{m} \min_{k \in \hat{S}} \| x_k^{(m)} \| \).

For the noiseless GMMV measurements (\( \lambda = 0 \)), with the unknown vectors being identical (\( x^{(1)} = \cdots = x^{(M)} \)), the recovery guarantee for SCP-GMMV can be simplified as follows.

**Corollary 2.** Under noiseless GMMV measurements, SCP-GMMV correctly recovers the sparsity pattern of the identical \( K \) block sparse vectors if,

\[
K < \frac{1}{2L} \left( 1 + \frac{1}{\bar{\gamma}} \right)
\]

where \( \bar{\gamma} \) denotes the average of the mutual incoherences of all sensing matrices, \( \bar{\gamma} = \frac{1}{M} \sum_{m=1}^{M} \gamma^{(m)} \).

**B. Remarks on SCP Recovery**

The noiseless recovery condition for the MMV model with identical sensing matrices given in (4) is the same as the recovery condition for single measurement vector case obtained in [19]. Note that the recovery guarantee in (4) must hold for every \( K \)-block sparse vectors including the case when \( x^{(1)} = \cdots = x^{(M)} \). In this special case of identical block sparse vectors, the corresponding measurements \( y^{(m)} \) are also identical (noiseless case). Hence, multiple measurements in MMV model do not provide new information and hence the recovery performance becomes identical to the SMV recovery. On the other hand, when the identical unknown vectors are sensed with different sensing matrices, the condition for the successful recovery given in (5) depends on the average of the mutual incoherence of the sensing matrices. Therefore, even if few of the sensing matrices individually fail to meet the required \( \gamma^{(m)} \) values, the perfect recovery is guaranteed as long as the average value \( \bar{\gamma} \) satisfies the requirement. Alternatively, the GMMV model observations (more precisely, signal component in the observations) do not become identical even when the unknown vectors are identical. Hence, SCP-GMMV recovery performance will be more robust to (any) correlation present in the unknown vectors while the SCP-MMV performance does not give much gains over SCP-SMV when the unknown vectors are highly correlated.

**C. Subspace Matching Pursuit**

Subspace matching pursuit (SMP) is also a greedy block sparse recovery algorithm proposed in [13]. We consider an extension of SMP for the GMMV model [23]. SMP-GMMV algorithm proceeds iteratively in the same manner as SCP-GMMV except that the selection of active sub-blocks is done based on the norm of the projection error of the residual. Specifically, the selection of active block of SMP-GMMV is done as

\[
q_k = \arg \max_{i = 1, \ldots, C} \sum_{m=1}^{M} \| \Pi_i^{(m)} (r_k^{(m)}) \|
\]

where \( \Pi_i^{(m)} = A_i^{(m)} \left( A_i^{(m)*} A_i^{(m)} \right)^{-1} A_i^{(m)*} \) denotes the projection matrix onto column space of \( A_i^{(m)} \). Rest of the steps are identical to SCP-GMMV. Mutual subspace incoherence of sensing matrices play a critical role in the recovery guarantee of SMP-GMMV.

**Definition 2.** The mutual subspace incoherence (\( \mu \)) of the sensing matrix \( A^{N \times CL} \) composed of sub-blocks \( A_i^{N \times L} \) is defined as,

\[
\mu = \max_{i, k \in \{1, \ldots, C\}} \left\{ \frac{\| \Pi_i^{(m)} (r_k^{(m)}) \|}{\| c_i \| \| c_k \|} \right\}.
\]

The value of \( \mu \) captures the smallest angle between column spaces of various sub-blocks. The recovery performance SMP-GMMV is given below with \( \mu^{(m)} \) being the mutual subspace incoherence of the sensing matrix \( A_i^{(m)} \). Towards that, for \( k \in S \), let us write \( A_k^{(m)} x_k^{(m)} = a_k^{(m)} v_k^{(m)} \) with \( v_k^{(m)} \in R(A_k^{(m)}) \)
and \( \|x_k^{(m)}\| = 1 \). The magnitude of the coefficients \( |a_k^{(m)}| \) play a role in the recovery performance.

**Theorem 2.** SMP-GMMV perfectly recovers the sparsity pattern of \( K \)-block sparse vectors from the observation model defined in (1) if,

\[
K < \min_{k \in S} \frac{1}{2} \left( 1 + \frac{\sum_{m=1}^{M} |a_k^{(m)}|}{\sum_{m=1}^{M} \mu^{(m)} |a_k^{(m)}|} \right) - \frac{M \lambda}{\sum_{m=1}^{M} \mu^{(m)} |a_k^{(m)}|}.
\]

(7)

The proof follows along the lines of Appendix A and is given in [23]. By defining \( \tilde{\mu} = \frac{1}{M} \sum_{m=1}^{M} \mu^{(m)} \), the recovery guarantees for the two special cases are given below.

**Corollary 3.** Using identical sensing matrices \( A^{(m)} = A \forall m \) with mutual subspace incoherence \( \mu \), SMP-MMV perfectly recovers the sparsity pattern of \( K \)-block sparse vectors if,

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\tilde{\mu}} \right) - \frac{1}{\tilde{\mu}} \frac{\lambda}{a_{\min}}.
\]

(8)

where \( a_{\min} = \min_{m} \min_{k \in S} |a_k^{(m)}| \).

**Corollary 4.** For the noiseless case with identical block sparse vectors \( x^{(1)} = \cdots = x^{(M)} \), SMP-GMMV correctly recovers the sparsity pattern if,

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\mu} \right).
\]

(9)

SMP recovery conditions are similar to the SCP conditions (with mutual subspace incoherence taking the role similar to the mutual incoherence). Hence, the impact of correlation among the unknown vectors on the SMP performance will be similar to that of SCP, which is discussed in Section III-B. In general, the block which gives maximum correlation with the residual (active block selection criterion for SCP) need not give the maximum norm for the projection of residual onto the column space of that block (active block selection criterion for SMP). Hence, the blocks recovered by SCP and SMP need not be always identical, resulting in differences in their recovery conditions.

**D. Single Step SMP**

Single step SMP (SSMP) algorithm for SMV presented in [19] chooses all the active blocks in a single step. SSMP-GMMV extends the idea to the GMMV model as follows.

**Step 1:** Compute \( \Gamma_i = \sum_{m=1}^{M} \| \Pi_i^{(m)} (y^{(m)}) \|_2 \) for \( i \in \{1, \ldots, C\} \).

**Step 2:** The sparsity pattern is detected as \( \hat{S} \subset \{1, \ldots, C\} \) with \( |\hat{S}| = K \) such that \( \Gamma_q > \Gamma_p, \forall q \in \hat{S}, \forall p \notin \hat{S} \).

Essentially, SSMP-GMMV projects the observations on each of the subspaces and chooses the \( K \) subspaces corresponding to top \( K \) values of the metric \( \Gamma_i \). Again, for \( k \in \hat{S} \), writing \( A_k^{(m)} x_k^{(m)} = a_k^{(m)} v_k^{(m)} \) with \( v_k^{(m)} \in \mathcal{R}(A_k^{(m)}) \), \( \|v_k^{(m)}\| = 1 \) and defining \( a_{\max} = \max_{k \in \hat{S}} |a_k^{(m)}| \) and \( a_{\min} = \min_{k \in \hat{S}} |a_k^{(m)}| \), we have the following recovery guarantee.

**Theorem 3.** SSMP-GMMV correctly recovers the sparsity pattern of \( K \)-block sparse vectors from the observation model defined in (1) if,

\[
K < \frac{1}{2} \left( 1 + \frac{\sum_{m=1}^{M} |a_{\min}^{(m)}|}{\sum_{m=1}^{M} a_{\max}^{(m)} |a_k^{(m)}|} \right) - \frac{M \lambda}{\sum_{m=1}^{M} a_{\max}^{(m)} |a_k^{(m)}|}.
\]

(10)

**Proof:** See Appendix B.

**Corollary 5.** With identical sensing matrices \( A^{(m)} = A \forall m \), SSMP-MMV perfectly recovers the sparsity pattern of \( K \)-block sparse vectors if,

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\mu a_{\min}} \right) - \frac{1}{\mu a_{\max}}.
\]

(11)

where \( a_{\min} = \min_{m} |a_k^{(m)}|, a_{\max} = \max_{m} |a_k^{(m)}| \).

**Corollary 6.** Under noiseless measurements, SSMP-GMMV correctly recovers the sparsity pattern of the identical \( K \) block sparse vectors if,

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\mu a_{\min}} \right).
\]

(12)

When compared with SMP-MMV, the recovery condition of SSMP-MMV is equivalent to amplifying \( \mu \) by a factor of \( \frac{\tilde{\mu}}{\mu} \), which is the penalty for reduction in complexity. However, when the \( \ell_2 \) norm of the non-zero blocks of \( x^{(m)} \) are equal, SSMP-MMV recovery condition is same as SMP-MMV. For the special case of \( L = 1 \), the SSMP-GMMV algorithm becomes identical to the single step OMP algorithm in [24], which was proposed for the recovery of joint (conventionally) sparse signals.

**E. Relaxed Norm Minimization**

In [22] relaxed norm minimization (RNM) is applied for obtaining a solution matrix with sparse rows for a given set of linear measurements. The RNM algorithm is extended for block sparse GMMV model as follows. Let us define the matrix \( U \) with its \((i, m)\)th entry being \( U_{i,m} = \| A_i^{(m)} x_v^{(m)} \| \) and the matrix \( F \) with its columns being \( f^{(m)} = y^{(m)} - A^{(m)} x_v^{(m)}, m = 1, \ldots, M \). Since the columns of each sub-block of \( A_i^{(m)} \) are linearly independent, \( U_{i,m} \) is zero if and only if \( x_v^{(m)} = 0 \). Hence the block sparse structure of \( x_v^{(m)} \) induces conventional sparse structure on the columns of the matrix \( U \). Also, since the block sparse vectors share the same sparsity pattern, matrix \( U \) has sparse non-zero rows. The relaxed norm minimization for GMMV is a convex optimization algorithm, which is formulated as,

\[
\min \| U \|_r \text{ subject to } \| F \|_r \leq \sigma
\]

(13)

where the relaxed norm \( \| U \|_r \) [22], is defined as \( \sum_{i=1}^{C} \max_{m} |U_{i,m}| \). Essentially, relaxed norm of a matrix is equivalent to applying \( \ell_\infty \) norm to each of the row of the matrix and applying \( \ell_1 \) norm on the resulting column vector. If the matrix \( U \) has a single column, relaxed norm minimization in (13) is equivalent to \( \ell_1 \) norm minimization. RNM-GMMV is identical to subspace base pursuit [13] for the single measurement vector case \( (M = 1) \).
From the solution \( \hat{x}^{(m)} \) obtained from the convex programming defined in (13), the support of the non-zero blocks \( \hat{S} \) is obtained as,

\[
\left\{ i \mid \sum_{m=1}^{M} \| \hat{x}^{(m)}_i \|_1 > \sum_{m=1}^{M} \| \hat{x}^{(m)}_j \|_1, \forall j \in \hat{S}, k \notin \hat{S}, |\hat{S}| = K \right\}.
\]

(14)

Essentially, the non-zero blocks are detected by choosing the subspaces corresponding to top \( K \) values of \( \sum_{m=1}^{M} \| \hat{x}^{(m)}_k \|_1 \).

Theorem 4. Under noiseless measurements \( y^{(m)} = A^{(m)} x^{(m)} \), the optimization problem defined in (13) correctly identifies the sparsity pattern of unknown \( K \) block sparse vectors if:

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\mu_{\text{max}}} \right)
\]

(15)

where \( \mu_{\text{max}} = \max_m \mu_m \).

Proof: See Appendix C.

The above recovery condition is governed by the worst case sensing matrix (the one with the highest value of \( \mu \)) and hence is very pessimistic. Obtaining better guarantee for convex programming based algorithms for GMMV models is an open problem.

F. Concatenated Block Sparse Signal Model

By defining \( y = [y^{(1)} \cdots y^{(M)}]^t \), \( x = [x^{(1)} \cdots x^{(M)}]^t \), we get the alternate model

\[
y = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 & \cdots & \hat{A}_C \end{bmatrix} \hat{x} + w,
\]

(16)

with \( \hat{A}_i = \begin{bmatrix} \hat{A}_i^{(1)} & 0 & \cdots & 0 \\
0 & \hat{A}_i^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{A}_i^{(M)} \end{bmatrix} \).

Here, the sensing matrix \( A \) consists of \( C \) blocks \( \{ \hat{A}_i \} \) each of size \( NM \times LM \) as defined in (16). Due to the joint block sparse structure of \( x^{(m)} \), the unknown vector \( x \) in (16), which is the concatenation of blocks \( \{ \hat{x}^{(i)} \} \) each of size \( ML \), has at most \( K \) non-zero blocks. Conventional block sparse algorithms can be employed to recover \( x \), and subsequently \( x^{(m)} \), using the model in (16). Now, we compare this alternate approach with our GMMV recovery algorithms.

Consider the block OMP (or SCP) algorithm from [12], [19]. Let \( r_k \) denote the residual obtained after \( k \) iterations, after removing the previously detected blocks (using least squares projection) from the observation \( y \). Similar to the construction of \( y \), we can interpret \( r_k = [r_k^{(1)} \cdots r_k^{(M)}]^t \). In the \( k^{th} \) iteration, block OMP chooses the block \( q_k = \arg \max_i \| A_i^* r_{k-1} \|_1 \). Due to the block diagonal structure of \( \hat{A}_i \), the block selection metric can be written as \( \| A_i^* r_{k-1} \|_1 = \sum_{m=1}^{M} \| A_i^{(m)}^* r_{k-1}^{(m)} \|_1 \). Clearly, this block OMP metric differs from the metric used by SCP-GMMV, which is \( \sum_{m=1}^{M} \| A_i^{(m)}^* r_{k-1}^{(m)} \|_1 \) and hence their recovery performances differ. It can be shown that (using the approach as in [19] and exploiting the block diagonal structure of \( A_i \)), the block OMP algorithm when applied to the model (16), will perfectly recover the sparsity pattern under noiseless conditions (\( \lambda = 0 \)), if \( K < \frac{\gamma_A}{2} \left( 1 + \frac{1}{\gamma_A} \right) \), where \( \gamma_A = \max_m \gamma_m^{(m)} \), the mutual incoherence of matrix \( A \) in (16). In comparison, with \( \lambda = 0 \) in (3), SCP-GMMV will perfectly recover the sparsity pattern if \( K < \frac{1}{2} \left( 1 + \frac{1}{\gamma'} \right) \) where \( \gamma' \leq \gamma_A \) (with strict inequality if not all \( \gamma_m^{(m)} \) are identical), the value of \( K \) guaranteed for perfect recovery from the block OMP algorithm is less than (or equal to) that of SCP-GMMV.

SMP algorithm [13] when applied to the concatenated model (which also differs from our SMP-GMMV algorithm), can perfectly recover the sparsity pattern under noiseless case, if \( K < \frac{1}{2} \left( 1 + \frac{1}{\mu_A} \right) \), where \( \mu_A \) is the mutual subspace incoherence of matrix \( A \). It can be shown that \( \mu_A \geq \max_m \mu_m^{(m)} \). Similarly, sparsity recovery condition of SSMP [19] algorithm is \( K < \frac{1}{2} \left( 1 + \frac{1}{\mu_{\text{min}}} \right) \), where \( \mu_{\text{min}}, \mu_{\text{max}} \) are as defined in (11). Again, the values of \( K \) guaranteed for perfect sparsity pattern recovery applying SMP and SSMP algorithms to the concatenated model is less than (or equal to) that of the noiseless SMP-GMMV and SSMP-GMMV conditions (7) and (10) respectively. Further, a convex programming based block sparse signal recovery algorithm, subspace base pursuit (SBP) [13], can be applied to the observation model in (16) and the successful recovery of sparsity pattern is guaranteed [13] if \( K < \frac{1}{2} \left( 1 + \frac{1}{\mu_A} \right) \). Since \( \mu_A \geq \max_m \mu_m^{(m)} \), RNM-GMMV guarantee (15) is better than that of SBP.

IV. CELL SEARCH IN LTE

In cellular systems, a mobile terminal must connect to a suitable base station in order to exchange information. Each BS has a unique identity and sends information in a sequence of frames. Before connecting with the BS, the MT needs to perform cell search, and recover the following parameters - the cell identity of the suitable BS, starting time of its frames (frame timing) and residual frequency offset (between the oscillators at MT and the corresponding BS). In this section, we establish that the cell search problem in heterogeneous cellular networks can be formulated as a joint block sparse signal recovery problem.

A. LTE frame structure

LTE employs OFDM based transmissions and has a frame duration of 10 ms. The training signals referred as synchronization signals are present in every frame in order to facilitate the cell search in MT. In each frame, synchronization signals are present at two locations, in preamble (PA) and in midamble
unique identity, chosen from the set \( \mathcal{I} = \{0, \cdots, 503\} \). This set of identities are divided into three groups \( \mathcal{G} = \{0, 1, 2\} \) and within each group, there are 168 identities from the set \( \mathcal{J} = \{0, \cdots, 167\} \) [25]. LTE synchronization signals, primary synchronization signal (PSS) and secondary synchronization signal (SSS), are chosen based on the identity of the corresponding BS and are transmitted in adjacent OFDM symbols.

PSS is chosen based on the cell group \( g_q \in \mathcal{G} \) while SSS is chosen based on the identity within a group \( s_q \in \mathcal{J} \) so that overall cell identity of BS is identified from PSS and SSS as \( q = 3s_q + g_q \in \mathcal{I} \). While PSS remains the same in both preamble and midamble, SSS differs as shown in Fig. 1.

![Fig. 1. LTE frames with PSS and SSS](image)

**B. Details of Synchronization Signals**

Since LTE is based on OFDM transmission technique, synchronization signals are specified in the frequency domain and the corresponding time domain signals are obtained by taking IFFT. In the frequency domain, PSS is composed of Zhadoff-Chu sequence

\[
\tilde{p}(k) = e^{-j\frac{\pi \mu k (k+1)}{64}} \quad \text{for} \quad k = 0, 1, \cdots, 63 \tag{17}
\]

where the root index \( \mu \) takes one of the values 25, 29, 34 based on the cell group \( g_q \in \mathcal{G} \). \( \{\tilde{p}(k)\} \) is mapped to the 62 tones around zero frequency (DC tone is punctured) and IFFT is taken to generate the time domain PSS \( \{p(n)\} \). SSS in frequency domain is obtained by concatenation and interleaving of two length-31 M-sequences which are unique to the identity \( s_q \in \mathcal{J} \) within a group. Moreover, these M-sequences are scrambled with another length-62 sequence which depends on cell group \( g_q \) and SSS changes between preamble and midamble. Hence there are a total of 1608 SSS sequences \( \{s_q\} \) as unique to the identity \( s_q \in \mathcal{J} \) within a group. Therefore, SSS sequence \( s_q^{pos}(n) \) and the location dependent SSS sequence \( s_q^{pos}(n) \) are represented as

\[
PSS : p_q(n) = \{p(n)g_q, n = 0, \cdots, 63\} \tag{18}
\]

\[
SSS : s_q^{pos}(n) = \{s(n)g_qr_q^{pos}, n = 0, \cdots, 63\} \tag{19}
\]

where \( g_q = q \mod 3, s_q = \lfloor \frac{q}{3} \rfloor \) and \( pos \in \{PA, MA\} \).

**C. HetNet Signal Model**

The observation at a MT is a superposition of signals from various BS in the neighborhood of MT. The discrete time base band model for the received observation \( \{y^{(r)}(n)\} \) at a mobile terminal at \( r^{th} \) receive antenna \( r \in \{0, \cdots, R - 1\} \) is given as,

\[
y^{(r)}(n) = \sum_{q \in B} \sum_{\ell = 0}^{L_q - 1} \sqrt{P_q} h_q^{(r)}(n; \ell)t_q(n - \ell)e^{j2\pi\epsilon_q n} + w^{(r)}(n),
\]

where \( B \subset \mathcal{I} \) denotes the set of cell identities of the BS in the neighborhood of MT, \( \{t_q(n)\} \) represents the signal transmitted by the BS with identity \( q \) with power level \( P_q \), \( \epsilon_q \in [-\epsilon_{\text{max}}, \epsilon_{\text{max}}] \) denotes the normalized relative frequency offset and \( w^{(r)}(n) \) denotes the additive noise. Channel impulse response (CIR) between \( r^{th} \) antenna of MT and BS with identity \( q \) at time instant \( n \) is denoted by \( \{h_q^{(r)}(n; \ell)\} \) with \( L_q \) being the corresponding delay spread. The number of neighbor base stations in the vicinity of the mobile terminal is bounded as \( |B| = K \leq K_{\text{max}} \) and the length of PSS/SSS is \( N = 64 \). We assume that the starting times of frames of all the BS in the neighborhood \( (B) \) are aligned, which is a typical assumption in the study of HetNets [21], [26]. See [19] for a detailed justification of this frame alignment assumption.

We parse the received observations into windows of size \( F \) where \( F \) denotes the number of samples between two adjacent PSS (such that synchronization signal is present in each window). Specifically, the received samples corresponding to the \( m^{th} \) observation window (half frame) and \( r^{th} \) receive antenna are given by \( \{y^{(m,r)}(n) = y^{(r)}(mF + n), n = 0, \cdots, F - 1\} \). Since the sample collection started at an arbitrary time (with respect to the frame starting time), the location of PSS can be any where within the half frame windows. However, the location of the PSS will be the same in all the observation windows. Based on the frame alignment of BS in \( B \), we assume that reception of PSS from all the BS falls within a time window. Specifically, with \( T_q \) being the time of reception of PSS from BS \( q \), i.e.,

\[
\{t_q(mF + n + T_q) = p_q(n), n = 0, \cdots, N - 1, \forall n\},
\]

we assume that \( [T_q, T_q + L_q] \subset [T, T + \Delta] \), \( \forall q \in B \), for some \( T \) and \( \Delta \). We assume that the neighbor set \( B \) remains unchanged during the reception of \( M \) half frames, for the typical \( M \) values of interest.

Due to the differences in power levels of different kinds of BS in HetNets, the interference becomes a key issue. In many cases, the MT may not connect to the BS with the strongest received power, since it may be a restricted access femto BS (See [19] for more details). In the HetNet cell search problem, the MT tries to recover the synchronization parameters (timing, frequency offsets and cell identity) of all the BS in its neighborhood \( (B) \) so that it can subsequently choose one of them suitable for connection. In the LTE cell search, (half) frame timing and cell group identity \( g_q \) is typically identified by correlating the received observations with PSS and the identity within a group \( s_q \) along with the preamble/midamble detection is done using SSS observations.

\( ^1 \Delta \) accounts for the different propagation delays from the neighbor BS.
D. Timing Detection Algorithm

By correlating the received samples with all three PSS sequences, we find the location of PSS in our collection of samples (half frames). Towards that, we define the $N$-length vectors $y_{k}^{(m,r)} = [y_{k}^{(m,r)}(k), \ldots, y_{k}^{(m,r)}(k + N - 1)]$, $k \in \{0, \ldots, F - 1\}$, and the matrix $\mathbf{P}_{g} \in \mathbb{C}^{N \times L}$ with its $\ell$th column being $\{y_{\ell}^{(m,r)}(n - \ell)\} = \{y_{\ell}^{(m,r)}(n - \ell)|n = 0\text{ to }N - 1\}$. Assuming CIR remains constant during the SSS transmission, we form the $L \times 1$ channel vectors $\mathbf{z}^{(m,r)} \forall q \in B$ (with zero padding done similar to the CIR construction for PSS duration). With $\mathbf{w}^{(m,r)}$ denoting the noise vector for SSS duration, we write the SSS observation vector as

$$
y^{(m,r)} = \sum_{q \in B} S_{q}^{y_{q}^{(m,r)}} \mathbf{z}^{(m,r)} + \mathbf{w}^{(m,r)} \quad (23)
$$

where $\{\mathbf{w}^{(m,r)}\}$ is the effective noise. With three different PSS sequences, we modify the joint correlator detector (JCD) discussed in [19] to find the timing using multiple observation windows as,

$$
T = \arg \max_{\ell \in \mathbb{G}} \sum_{q \in B} \sum_{k \in \{0, 1, \ldots, F - 1\}} \left\| \mathbf{P}_{g} y_{k}^{(m,r)} \right\|^{2} \quad (22)
$$

where $\mathbb{G}$ denotes the power set of $\mathbb{G}$. Essentially, we perform moving window correlation of received samples (by varying the index $k$) over all possible combinations of PSS sequences (using the power set $\mathbb{G}$)) and find which location and which combination of PSS sequences gives maximum correlation. Due to the overlap of PSS from multiple BS, we consider the correlation metric over all possible combinations of superimposed PSS. Under some additional assumptions, JCD has been shown to be identical to the ML detector in [19].

E. Recovering Cell Search Parameters

In this section, we establish that the recovery of the remaining cell search parameters $\mathbf{s}_{q}, \mathbf{e}_{q}, \forall q \in B$ can be solved using joint block sparse recovery formulation. After the location of PSS is obtained by timing detection, we can collect the received samples corresponding to the SSS from the observations. Let $\mathbf{y}^{(m,r)}$ denote the $N \times 1$ vector of SSS observations from the $m\text{th}$ half frame and $r\text{th}$ receive antenna. We write the SSS observations using matrix/vector quantities as follows. For $q \in L$ and pos $\in \{PA, MA\}$, we define the matrices $S_{\ell}^{y_{q}^{(m,r)}} \in \mathbb{C}^{N \times L}$ with its $\ell$th column being $\{s_{\ell}^{y_{q}^{(m,r)}}(n - \ell)|n = 0\text{ to }N - 1\}$. Assuming CIR remains constant during the SSS transmission, we form the $L \times 1$ channel vectors $\mathbf{z}_{\ell}^{(m,r)} \forall q \in B$ (with zero padding done similar to the CIR construction for PSS duration). With $\mathbf{w}^{(m,r)}$ denoting the noise vector for SSS duration, we write the SSS observation vector as

$$
y^{(m,r)} = \sum_{q \in B} S_{\ell}^{y_{q}^{(m,r)}} \mathbf{z}_{\ell}^{(m,r)} + \mathbf{w}^{(m,r)} \quad (23)
$$

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$$
T = \arg \max_{\ell \in \mathbb{G}} \sum_{q \in B} \sum_{k \in \{0, 1, \ldots, F - 1\}} \left\| \mathbf{P}_{g} y_{k}^{(m,r)} \right\|^{2} \quad (22)
$$

where $\mathbb{G}$ denotes the power set of $\mathbb{G}$. Essentially, we perform moving window correlation of received samples (by varying the index $k$) over all possible combinations of PSS sequences (using the power set $\mathbb{G}$)) and find which location and which combination of PSS sequences gives maximum correlation. Due to the overlap of PSS from multiple BS, we consider the correlation metric over all possible combinations of superimposed PSS. Under some additional assumptions, JCD has been shown to be identical to the ML detector in [19].

2The CIR vector $\mathbf{h}_{q}^{(m,r)}$ has the same length $L$ for all $q \in B$ with suitable zero-paddingle (accounting for different propagation delays and different delay spreads of the BS in $B$). We assume that $L$ is smaller than the cyclic prefix length. More details are given in [19].
F. GMMV Recovery Performance in LTE Cell Search

In this section, we consider reformulation of the recovery guarantees of the greedy and single-step algorithms discussed in Section III. The set of active blocks in $A^{(m)}$ which contribute to the observation in (25) is given by $\{S_{q,\text{pos}}^{(m)}, q \in B\}$. While theorems in Section III gives recovery guarantees for any set of active blocks, we now consider the recovery guarantees for a specific set of active blocks. Note that, these conditions will depend on the correlation of those specific active blocks with the rest of the blocks. Towards that, we define the parameters $\gamma_{B}^{(m)}$ and $\mu_{B}^{(m)}$ as follows,

$$\gamma_{B}^{(m)} = \max_{r_i \neq r_k} \left\{ \|r_i^\top r_k\| \right\}$$

$$\mu_{B}^{(m)} = \max_{A_r \neq A_s} \left\{ \max_{u, w \in \mathbb{C}^L} \|A_r u\|_2 \|A_s w\|_2 \right\}$$

with $r_i$ being any column of active blocks $\{S_{q,\text{pos}}^{(m)}, q \in B\}$ and $r_k$ being any column of $A^{(m)}$ (from active or non-active blocks) but not identical to $r_i$, and

$$\gamma_{B}^{(m)} = \max_{r_i \neq r_k} \left\{ \|r_i^\top r_k\| \right\}$$

with $A_r \in \mathbb{C}^{N \times L}$ being an active block from the set $\{S_{q,\text{pos}}^{(m)}, q \in B\}$ and $A_b \in \mathbb{C}^{N \times L}$ being any block from $A^{(m)}$ (which is not identical to $A_r$). The following lemma characterizes the performance of SCP-GMMV in recovering the sparsity pattern corresponding to a specific set of active blocks. The proof is similar to that of Theorem 1.

Lemma 1. Given the GMMV model (25) with the specific set of active blocks in $A^{(m)}$ being $\{S_{q,\text{pos}}^{(m)}, q \in B\}$, SCP-GMMV perfectly recovers the cell search parameters of the neighbor base stations if, $K < \frac{1}{2L} \left( 1 + \frac{\sum_{m,r} \|\sigma_{q,\text{pos}}^{(m)}\|_{\gamma_{B}^{(m)}}}{\sum_{m,r} \gamma_{B}^{(m)} \|z_q^{(m,r)}\|} \right) - \frac{MRL}{\sqrt{L} \sum_{m,r} \gamma_{B}^{(m)} \|z_q^{(m,r)}\|}$

Similarly, with the same set of active blocks, the conditions for successful recovery of the cell search parameters for SMP-GMMV and SSMP-GMMV, are given respectively as,

$$K < \frac{1}{2} \left( 1 + \frac{\sum_{m,r} a_{q,\text{pos}}^{(m,r)}}{\sum_{m,r} \mu_{B}^{(m)} a_q^{(m,r)}} \right) - \frac{MRL}{\sum_{m,r} \mu_{B}^{(m)} a_q^{(m,r)}},$$

$$K < \frac{1}{2} \left( 1 + \frac{\sum_{m,r} a_{q,\text{pos}}^{(m,r)}}{\sum_{m,r} \mu_{B}^{(m)} a_q^{(m,r)}} \right) - \frac{MRL}{\sum_{m,r} \mu_{B}^{(m)} a_q^{(m,r)}},$$

where $a_{q,\text{pos}}^{(m,r)} = \|S_{q,\text{pos}}^{(m)}\|z_q^{(m,r)}\|$, $a_q^{(m,r)} = \max_{q \in B} a_{q,\text{pos}}^{(m,r)}$, and $a_q^{(m,r)} = \min_{q \in B} a_{q,\text{pos}}^{(m,r)}$. Using these conditions, we establish connections between the analytical recovery guarantees and the LTE cell search simulation performance in Section V-C.

G. Computational Complexity

In this section, we discuss the computational complexity of various recovery algorithms to the cell search problem in LTE. Relaxed norm minimization (RNM) can be solved using second order cone programming in polynomial time complexity of $O(\sqrt{DC})$ [28]. For the other algorithms, we measure the real-time complexity in terms of the number of multiplications performed in running the algorithm. The greedy algorithms SCP-GMMV and SMP-GMMV incur most of the complexity at Step 2 for finding an active block and at Step 5 for computing the residual. The projection matrices $\Pi_t^{(m)}$ required in SMP-GMMV and SSMP-GMMV can be pre-computed and stored at the receiver. Table I provides the comparison between the complexities of our proposed algorithms and the conventional successive interference cancellation (SIC) [29] for LTE cell search.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>No. of Multiplications for $K$ iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCP</td>
<td>$KMDCNL + MN(K^2L^2 + K^2L) + O(L^2)$</td>
</tr>
<tr>
<td>SMP</td>
<td>$KMDCN^2 + MN(K^2L^2 + K^2L) + O(L^2)$</td>
</tr>
<tr>
<td>SSMP</td>
<td>$DCN^*$</td>
</tr>
<tr>
<td>SIC [29]</td>
<td>$KMDCNL + KMN^2$</td>
</tr>
</tbody>
</table>

Table I: Computational Complexity Comparison.

V. Simulation Results

In this section, using Monte-Carlo simulations, we study the performances of the joint block sparse recovery algorithms described in Section III with random sensing matrices as well as the LTE cell search models (25).

A. Random Sensing Matrices

We generate the sensing matrices $A^{(m)}$ randomly with their entries being i.i.d normal distribution. For MMV model, we generate one such random matrix and use it to generate all the observations $\{y_q^{(m)}\}$, while for the GMMV model, we independently generate $M$ different sensing matrices. We use white Gaussian noise model for the additive noise. For any given $m$, the non-zero entries in $x^{(m)}$ are generated independently. However, across $m$, the non-zero entries are generated using Gauss-Markov model to introduce the correlation. Let $\rho$ denote the correlation between $x^{(m)}$ and $x^{(m+1)}$, such that $\rho = 1$ implies all the unknown vectors are identical and $\rho = 0$ represents that the non-zero entries are uncorrelated across $m$.

In Fig. 2, we compare the sparsity pattern recovery performance of joint sparse recovery algorithms SCP, SMP, SSMP and RNM for both GMMV and MMV models. We also compare the performance of their SMV counterparts which recover the $M$ sparse vectors $x^{(m)}$ separately, ignoring the fact that they all share the same sparsity pattern. Clearly joint sparse recovery methods perform better than SMV. Performance of SMV degrades with $M$ since the success is declared only if the recovered sparsity pattern is correct for all the $M$ unknown vectors. We also see that convex programming algorithm RNM has the best recovery performance followed by SMP, SCP and SSMP, clearly indicating the performance versus complexity tradeoff.

In Fig. 3, we study the performances of RNM and SCP for various values of $\rho$. With increase in $\rho$, SCP-MMV recovery
performance gets poorer. For $\rho = 1$, SCP-MMV performs close to SCP-SMV. On the other hand, the SCP-GMMV recovery is robust to the the value of $\rho$. The impact of $\rho$ on the SCP numerical results are in accordance with our inferences discussed in Section III-B. Interestingly, RNM also exhibits a similar performance with respect to the correlation $\rho$.

For $\rho = 1$, SCP-MMV performs close to SCP-SMV. On the other hand, the SCP-GMMV recovery is robust to the value of $\rho$. The impact of $\rho$ on the SCP numerical results are in accordance with our inferences discussed in Section III-B. Interestingly, RNM also exhibits a similar performance with respect to the correlation $\rho$.

![Fig. 2. GMMV vs MMV model $C = 50$, $L = 5$, $K = 6$, $N = 64$, $\rho = 0.5$.](image1)

![Fig. 3. GMMV vs MMV model $C = 50$, $L = 5$, $M = 3$, $N = 64$.](image2)

### B. LTE Cell Search

For the LTE cell search simulations, we adopt the following set-up. The additive noise is Gaussian and the CIR of the neighbor BS are generated using extended typical urban model (ETU) and pedestrian A model [30] for macro BS and low power BS respectively. The Jake’s model is used to generate the time-varying channel to incorporate the mobility. The number of receive antennas is set as $R = 2$ as per the LTE specification and CIRs of those two antennas are uncorrelated. The frequency offset $\epsilon_f$ is uniformly distributed between $\pm 7$ppm with 2 GHz carrier frequency.

The cell search algorithms detect the frame timing as per (22) and the other synchronization parameters (identities in $B$, frequency offset bin indices $\{i_q\}$ and preamble/midamble information) are extracted from (25) using the joint block sparse recovery framework. For comparison, we also study the performance of the successive interference cancellation (SIC) discussed in [29], with necessary modifications to accommodate frequency offsets. The number of frequency bins is set as $D = 5$. Since the exact size of $B$ is not known at the MT, we introduce stopping criterions [19] by setting thresholds for residuals in the joint block sparse recovery algorithms. In order to compare fairly, the threshold for each algorithm is set such that the average number of falsely detected cells satisfy the constraint $\mathbb{E}[|B \setminus B|] < 1$. We declare successful synchronization if, the frame timing, cell identity and the closest frequency offset bin are perfectly detected for all the BS in the neighbor set. The detection performances are plotted for various SNR values where the signal power is the sum of received power from all the neighbor BS.

The Fig. 4 shows the successful synchronization performance with respect to SNR of our proposed recovery algorithms. It can be seen that the joint sparse recovery algorithms perform better than the SIC algorithm proposed in [29].

![Fig. 4. Performance of different recovery algorithms ($K = 6$).](image3)

![Fig. 5 shows the synchronization performance of the algorithms for different number of neighbor BS. Greedy algorithms can not recover the blocks if $KL > N$ due to the inability to solve the least squares minimization (Step 5 in SCP). However, the convex programming based RNM can recover the support for higher values of $K$, but the reconstructed non zero blocks (CIRs) will be inaccurate.](image4)

Fig. 6 shows the recovery performances for two different speeds of the MT. As the speed increases, the temporal correlation among the CIRs across multiple observations approaches zero and the recovery algorithms perform better. However in practice, it should be noted that if the MT is moving very fast, the number of half frames ($M$) over which the neighbor set $B$ remains unchanged will be small.

For Fig. 7, we set one of the BS in $B$ as the desired target BS, treating the rest of the neighbor BS as interferers. Fig. 7 shows the average delay until the synchronization parameters are successfully acquired for the target BS for various values of signal to interference plus noise ratio (SINR). Recall that one half frame window (containing preamble or midamble) occupies a duration of 5 ms. We see that our algorithms
perform well even at low SINR values and are superior to the SIC based cell search algorithm from [29]. We also see that the SMV algorithms which tries to recover the synchronization parameters separately with each half frame (discarding the past observations corresponding to recovery failures) incur significantly longer delays than our GMMV algorithms which perform joint recovery (using all the previous half frames).

While we have previously done timing detection using PSS and recovering the rest of the cell search parameters using SSS, it is also possible to look at the concatenation of PSS and SSS as a single sequence (with 1008 possibilities) and perform the cell search using this concatenated sequence. Note that computing the timing detection with this concatenated sequence will incur substantial increase in complexity. A comparison between the above mentioned concatenated (with label C) and the previous decoupled (with label D) approach in acquiring a target BS is shown in Fig. 8. It can be seen that the recovery performances improve in concatenated approach but with increased computational complexity.

C. Connection between Analytical and Simulation Results

In this section, we establish the connection between the simulation performance and the analytical recovery conditions

![Fig. 5. Performance with different number of neighbor BS (SNR=10dB).](image)

![Fig. 6. Performance comparison with different Dopplers $K = 6$, SNR=5dB.](image)

![Fig. 7. Delay to correctly synchronize with a target base station ($K = 6$).](image)

![Fig. 8. Detection performance using concatenated PSS and SSS ($M = 3$, $K = 6$).](image)

![Fig. 9. Performance in LTE with respect to the number of neighbor BS.](image)
of our greedy and single-step algorithms in the context of LTE cell search. For the cell search, timing detection is done (commonly) as per (22) and the recovery of neighbor set parameters is obtained using simulations and analytical recovery conditions (separately) as follows. With LTE frame structure, \( D = 5 \) frequency bins and \( M = 3 \) half-frame observation windows, we obtain the simulation performance for the successful synchronization by varying the number of neighbor BS \( K \). These results are shown in Fig. 9 with label Sim. In order to evaluate the recovery performance using the analytical recovery conditions, we first construct the LTE sensing matrices \( A^{(0)} \) and \( A^{(1)} \) from (25). Then we compute the values of \( \gamma_B^{(m)} \), \( m = \{0, 1\} \) and \( \mu_B^{(m)} \), \( m = \{0, 1\} \) defined in (27) and (28), for each possible set of active BS \( B \). The values of \( \gamma_B^{(m)} \) range between 0.01 to 0.42 and \( \mu_B^{(m)} \) range between 0.034 to 0.67, depending on the active neighbor BS set \( B \). Note that, when the set of neighbor BS \( B \) is chosen randomly (with uniform distribution), the quantities \( \gamma_B^{(m)} \), \( \mu_B^{(m)} \) and the fading channel coefficients \( z_{(m,r)}^{n} \) appearing in the right hand side (RHS) of the recovery conditions (29), (30) and (31) are random. Using Monte-Carlo simulations (by averaging over the fading channel coefficients and the set of active neighbor BS), for each value of \( K \), we evaluate the probabilities that the chosen value of \( K \) is less than the RHS of conditions (29), (30) and (31). These probability values are plotted under the label Theory in Fig. 9. As can be seen in Fig. 9, plots obtained from our analytical recovery conditions are reasonably good in estimating the recovery performance of these algorithms and their relative performances. We also note that the simulation performance is slightly better than that of the theory, which is a common trend in the compressive sensing literature [8].

**D. Comparison between GMMV and Concatenated Block Sparse Recovery**

In Fig. 10, we compare the numerical performance of GMMV recovery algorithms (prefixed with ‘GMMV’) and the block sparse recovery algorithms (prefixed with ‘block’) applied to the concatenated model (16), for both random sensing matrices (with \( C = 50 \), \( K = 6 \), \( L = 5 \), \( M = 4 \)) and in the context of LTE synchronization (with parameters same as Fig. 4). In accordance with theoretical recovery guarantees from Section III-F, the GMMV recovery algorithms developed in this work perform better than the corresponding block sparse signal recovery algorithms applied for the concatenated model.

**VI. CONCLUSION**

In this paper, we considered the GMMV block sparse signal recovery problem and studied greedy and convex programming based algorithms. As a special case, we applied our results to the MMV model and presented how the correlation among block sparse signals affect the recovery performance. Specifically, we showed that, while the MMV model is sensitive to the correlation, the GMMV model is more robust. Next, we discussed the cell search problem in LTE HetNets and showed that the cell search with observations from multiple antennas and/or frames can be solved using GMMV block sparse signal recovery framework. Simulation results showed that the cell search algorithms proposed using our framework perform significantly better than the successive interference cancellation algorithm.

**APPENDIX**

**A. Recovery Guarantee for SCP-GMMV**

Without loss of generality (WLOG), we assume that the first \( K \) blocks of each vector \( x^{(m)} \) are non-zero. Specifically, we write the observation vectors (1) as \( y^{(m)} = \tilde{y}^{(m)} + w^{(m)} \), with the noiseless component being

\[
\tilde{y}^{(m)} = \sum_{i=1}^{K} a_i^{(m)} w_i^{(m)},
\]

where \( w_i^{(m)} \in \mathcal{R}(A_i^{(m)}) \) with \( \|w_i^{(m)}\| = 1 \). We also WLOG assume that \( |a_1^{(m)}| \geq \cdots \geq |a_K^{(m)}|, |x_1| \geq \cdots \geq |x_K| \).

The algorithm will choose the correct subspace in the first iteration if,

\[
\sum_{m=1}^{M} \left\| A_i^{(m)} \right\| y^{(m)} \left\| >_t \right. \sum_{m=1}^{M} \left\| A_i^{(m)} \right\| y^{(m)} \left\| \quad \forall t > K. \right.
\]

We bound the above terms as follows.

\[
\left\| A_i^{(m)} \right\| y^{(m)} \left\| \geq \left\| A_i^{(m)} \right\| \tilde{y}^{(m)} \left\| - \left\| A_i^{(m)} \right\| w^{(m)} \left\|ight.ight.
\]

\[
\left\| A_i^{(m)} \right\| y^{(m)} \left\| = \left\| A_i^{(m)} A_i^{(m)} x_1 \right\| + \cdots + \left\| A_i^{(m)} A_K^{(m)} x_K \right\|ight.
\]

\[
\geq \left\| A_i^{(m)} A_i^{(m)} x_1 \right\| - \sum_{k=2}^{K} \left\| A_i^{(m)} A_k^{(m)} x_k \right\|ight.
\]

From Rayleigh Ritz principle [31],

\[
\nu_{\min} \left( A_i^{(m)} A_i^{(m)} \right) \left\| x_k \right\| \leq \left\| A_i^{(m)} A_k^{(m)} x_k \right\| \leq \nu_{\max} \left( A_i^{(m)} A_i^{(m)} \right) \left\| x_k \right\|,
\]

where \( \nu_{\min} \) and \( \nu_{\max} \) are the minimum and maximum eigen values respectively. By Gershgorin circle theorem [31] and bound on the inner products between the columns of \( A^{(m)} \) defined in Definition 1,
it can be verified that, $|\nu_{\min}(A_i^{(m)^*} A_i^{(m)})| \geq (1 - (L - 1)\gamma^{(m)})$, $|\nu_{\max}(A_i^{(m)^*} A_k^{(m)})| \leq L\gamma^{(m)}$, $\forall i, k \in \{1, \cdots, C\}, i \neq k$. Therefore,

$$\|A_i^{(m)^*} y^{(m)}\| \geq (1 - KL\gamma^{(m)} + \gamma^{(m)})\|x_1^{(m)}\|. \quad (36)$$

Now, for $t > K$, we have

$$\|A_i^{(m)^*} y^{(m)}\| \leq \|A_i^{(m)^*} y^{(m)}\| + \|A_i^{(m)^*} w^{(m)}\| \quad (37)$$

$$\|A_i^{(m)^*} y^{(m)}\| \leq \sum_{i=1}^K \|A_i^{(m)^*} x_i^{(m)}\| \leq KL\gamma^{(m)}\|x_1^{(m)}\|. \quad (38)$$

Using the facts that the columns of $A^{(m)}$ have unit norm and hence the norm of the inner product of any column with a sequence cannot exceed norm of the sequence itself, the bound on correlation of noise with any subspace becomes

$$\|A_i^{(m)^*} w^{(m)}\| \leq \sqrt{L}\lambda. \quad (39)$$

Using (34), (36), (37), (38) and (39) we get,

$$\sum_{m=1}^M \|A_i^{(m)^*} y^{(m)}\| \geq \sum_{m=1}^M \|x_1^{(m)}\|(1 + \gamma^{(m)} - KL\gamma^{(m)}) - M\sqrt{L}\lambda$$

$$\sum_{m=1}^M \|A_i^{(m)^*} y^{(m)}\| \leq \sum_{m=1}^M \|x_1^{(m)}\|KL\gamma^{(m)} + M\sqrt{L}\lambda. \quad (40)$$

Hence in the first iteration correct subspace is chosen if, $K < \frac{1}{2L} \left(1 + \frac{M}{M} \|x_1^{(m)}\| \right) - \frac{M\lambda}{\sum_{m=1}^M \sqrt{L}\gamma^{(m)}\|x_1^{(m)}\|}$. After each iteration, we project the residual onto the orthogonal complement of previously chosen blocks (column spaces) and hence we can repeat the above argument for every iteration and support of all the $K$ blocks are correctly recovered if (3) is satisfied.

**B. Recovery Guarantee for SSMP-GMMV**

WLOG we assume that first $K$ blocks in $x^{(m)}$ are active as done in Appendix A. For vectors $e^{(m)}$ in the null space of $A^{(m)}$, we know that $x^{(m)} + e^{(m)}$ are also solutions for $y^{(m)} = A^{(m)}x^{(m)}$. If the condition (15) is satisfied, then for any $e^{(m)}$, we establish that $\|U\|_{\infty} \leq \|U^{E}\|_{\infty}$ where the entries of $U, U^{E}$ being $U_{i,m} = \|A_i^{(m)}x_i^{(m)}\|$, $U^{E}_{i,m} = \|A_i^{(m)}(x_i^{(m)} + e_i^{(m)})\|$ respectively, and thus the minimum relaxed norm solution will correspond to $x^{(m)}$. Towards that, we first establish that, if (15) is satisfied, for any $e^{(m)}$ with $A^{(m)}e^{(m)} = 0$, we have

$$\sum_{i=1}^K \max_{m} \|A_i^{(m)} e_i^{(m)}\| < \sum_{i=K+1}^C \max_{m} \|A_i^{(m)} e_i^{(m)}\|. \quad (45)$$

Since $A^{(m)}e^{(m)} = 0$, defining $n_{i,m} = \|A_i^{(m)} e_i^{(m)}\|$ and $\Psi_{i,m} = \frac{1}{n_{i,m}}A_i^{(m)} e_i^{(m)}$, we have

$$\sum_{i=1}^C n_{i,m} \Psi_{i,m} = 0. \quad (46)$$

C. Recovery Guarantee for RNM-GMMV

We consider the noiseless measurements $y^{(m)} = A^{(m)}x^{(m)}$. WLOG we assume that first $K$ blocks in $x^{(m)}$ are active as done in Appendix A. For vectors $e^{(m)}$ in the null space of $A^{(m)}$, we know that $x^{(m)} + e^{(m)}$ are also solutions for $y^{(m)} = A^{(m)}x^{(m)}$. If the condition (15) is satisfied, then for any $e^{(m)}$, we establish that $\|U\|_{\infty} \leq \|U^{E}\|_{\infty}$ where the entries of $U, U^{E}$ being $U_{i,m} = \|A_i^{(m)}x_i^{(m)}\|$, $U^{E}_{i,m} = \|A_i^{(m)}(x_i^{(m)} + e_i^{(m)})\|$ respectively, and thus the minimum relaxed norm solution will correspond to $x^{(m)}$. Towards that, we first establish that, if (15) is satisfied, for any $e^{(m)}$ with $A^{(m)}e^{(m)} = 0$, we have

$$\sum_{i=1}^K \max_{m} \|A_i^{(m)} e_i^{(m)}\| < \sum_{i=K+1}^C \max_{m} \|A_i^{(m)} e_i^{(m)}\|. \quad (45)$$

Since $A^{(m)}e^{(m)} = 0$, defining $n_{i,m} = \|A_i^{(m)} e_i^{(m)}\|$ and $\Psi_{i,m} = \frac{1}{n_{i,m}}A_i^{(m)} e_i^{(m)}$, we have

$$\sum_{i=1}^C n_{i,m} \Psi_{i,m} = 0. \quad (46)$$
Pre-multiplying the above equation by $\Psi_{k,m}^*$, we get,
\[ n_{k,m} + \sum_{i=1}^{C} n_{i,m} \Psi_{k,m}^* \Psi_{i,m} = 0. \] (47)

Each vector $\Psi_{i,m}$ has unit norm and belongs to different column subspaces $A_i^m$, which have mutual subspace incoherence $\mu_i^m$. So, $|\Psi_{k,m}^* \Psi_{i,m}| \leq \mu_i^m$ for $i \neq k$ and hence $n_{k,m} \leq \mu_i^m \sum_{i=1}^{C} n_{i,m}$. If $K$ satisfies (15), we have
\[ 2 \sum_{i=1}^{K} \max_{m} \| A_i^m e_i^m \| \leq 2 M \frac{\mu_i^m}{1 + \mu_i^m} \sum_{i=1}^{C} \| A_i^m e_i^m \| \leq \sum_{i=1}^{C} \max_{m} \| A_i^m e_i^m \|, \] (48)
which is equivalent to (45). Now, we have
\[ \| U \|_x = \sum_{i=1}^{K} \max_{m} \| A_i^m x_i^m \| \leq \sum_{i=1}^{K} \max_{m} \| A_i^m (x_i^m + e_i^m) \| + \sum_{i=1}^{K} \max_{m} \| A_i^m e_i^m \| \] (49)
\[ < \sum_{i=1}^{K} \max_{m} \| A_i^m (x_i^m + e_i^m) \| + \sum_{i=K+1}^{C} \max_{m} \| A_i^m e_i^m \| \] (50)
\[ = \sum_{i=1}^{C} \max_{m} \| A_i^m (x_i^m + e_i^m) \| = \| U^E \|_x, \] (51)
where, (49) is triangle inequality, (50) is the application of (45) and (51) follows from the fact that only the first $K$ blocks of $x_i^m$ are non-zero. Hence, we have established that the minimum relaxed norm solution corresponds to the true $K$-block sparse solution if the condition (15) is satisfied.

REFERENCES