EE 6340: Information Theory
Final exam - May 2012

1. Let $X_1, Z_1, Z_2, \cdots, Z_n$ be i.i.d. fair binary random variables taking values 0 or 1. Now, define the sequence of random variables X_i as

$$X_{i+1} = X_i + Z_i, \quad i = 1, 2, \cdots, n-1,$$

where + denotes *real* addition. Show that the $\mathrm{I}(X_1; X_2, X_3, \cdots, X_n) = 1/2$. [7 pts]

2. Let X, Y, Z be fair binary random variables with the constraint that $\mathrm{I}(X;Y) = \mathrm{I}(Y;Z) = \mathrm{I}(X;Z) = 0$. Under this contraint, what is the minimum value of $\mathrm{H}(X,Y,Z)$? Give an example which achieves this minimum. [5 pts]

3. Let X be an exponential random variable with the probability density function

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (a) Find the differential entropy $h(X)$. [2 pts]

   (b) Find a function $g(\cdot)$ such that if $Y = g(X)$ then $h(Y) = 0$. [3 pts]

4. Consider two discrete memoryless channels $(\mathcal{X}, p_1(y|x), \mathcal{Y})$ and $(\mathcal{Y}, p_2(y|x), \mathcal{Z})$ with input-output alphabets $\{0, 1, e\}$. The transition probabilities are given below:

$X \setminus Y$	1	e	0
1	0	1	0
e	0	1	0
0	0	0	1

$p_1(y|x)$

$Y \setminus Z$	1	e	0
1	1	0	0
e	0	1	0
0	0	1	0

$p_2(z|y)$

   (a) Find the capacity C_1 of first channel with input X, output Y and transition probabilites $p_1(y|x)$. [3 pts]

   (b) Find the capacity C_2 of second channel with input Y, output Z and transition probabilites $p_2(z|y)$. [2 pts]

(c) Now, we cascade the two channels together so that input is X, output is Z and the transition probabilities are $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. Find the capacity C_3 of the cascaded channel. [3 pts]

(d) Now, we allow processing *in between* the two cascaded channels. Say we are allowed to *decode* the output sequence Y^n of first channel, re-encode and send a different sequence $\tilde{Y}^n$ thru the second channel. What is the cascaded channel capacity in this case? (Think $W \to X^n(W) \to Y^n \to \tilde{Y}^n(Y^n) \to Z^n \to \hat{W}$) [4 pts]

5. Consider a source of M symbols with probabilitites $p_1, p_2, \cdots, p_M$. Suppose that probabilities are arranged in descending order so that $p_1 \geq p_2 \geq \cdots \geq p_M$. We use binary Huffman code to encode the source.

   (a) Show that if the most probable symbol has probability $p_1 > 2/5$, then that symbol should be assinged a codeword of length 1. [4 pts]

   (b) Show that if the most probable symbol has probability $p_1 < 1/3$, then that symbol should be assigned a codeword of length ≥ 2. [3 pts]

6. Consider the two outputs Y_1 and Y_2 corresponding to a single input X as

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

where Z_1 and Z_2 are independent Gaussian with variances σ_1^2 and σ_2^2 respectively. Input power constraint is $\mathrm{E}\{X^2\} \leq P$.

   (a) Suppose that the receiver can view both outputs Y_1 and Y_2, what is the channel capacity? What type of signalling achieve capacity? [3 pts]

   (b) Suppose that the receiver can view only the sum $Y = Y_1 + Y_2$, what is the channel capacity? What type of signalling achieve capacity? [2 pts]

   (c) Suppose that the receiver can view only the weighted sum $\bar{Y} = \alpha Y_1 + (1-\alpha)Y_2$ where $0 \leq \alpha \leq 1$. Can you design α such that the channel capacity in this case is same as that of part (a). [4 pts]

7. Consider the following parallel Gaussian channel with 4 subchannels

$$Y_k = kX_k + Z_k, \quad k = 0, 1, 2, 3,$$

where Z_k's are i.i.d. white Gaussian noise with unit variance. Let power allotted to k^{th} subchannel be $P_k = \mathrm{E}\{X_k^2\}$. We have the total input power constraint $\sum_k P_k \leq P$.

   (a) Find the optimal power levels P_k to maximize the parallel channel capacity when total power $P = 3/2$. [2 pts]

   (b) Find the maximum value of power level P_c so that whenever $P < P_c$, optimal policy uses only one of the subchannels. [3 pts]