Adaptive Signal Processing

Assignment - I: Solutions

1.6 Biased Measurements

$$y = x + v$$

$$x = \pm \text{ with equal Probability}$$

$$v \sim N(\bar{v}, 1)$$

The least mean square estimator of x given y:

$$\hat{x} = E[x \mid y = y]$$

$$= 1 \cdot P[x=1 \mid y=y] - 1 \cdot P[x=-1 \mid y=y] \quad \text{—(1)}$$

$$P[x=1 \mid y=y] = \frac{f(y \mid x=1) \cdot P(x=1)}{f(y)}$$

$$P(x=-1 \mid y=y) = \frac{f(y \mid x=-1) \cdot P(x=-1)}{f(y)}$$

$$f(y \mid x=1) = \frac{1}{\sqrt{2\pi}} e^{-(y-(\bar{v}+1))^2/2}$$

$$f(y \mid x=-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[y-(\bar{v}-1)]^2}$$

$$f(y) = \tfrac{1}{2} \cdot [f(y \mid x=1) + f(y \mid x=-1)]$$

$$P(x=1) = P(x=-1) = \tfrac{1}{2}.$$

Substituting these in (1)

$$\hat{x} = \frac{e^{-\frac{[y-(\hat{v}+1)]^2}{2}} - e^{-\frac{[y-(\hat{v}-1)]^2}{2}}}{e^{-\frac{[y-(\hat{v}+1)]^2}{2}} + e^{-\frac{[y-(\hat{v}-1)]^2}{2}}}$$

$$= \frac{e^{-(\hat{v}-y)} - e^{-(y-\hat{v})}}{e^{-(\hat{v}-y)} + e^{-(y-\hat{v})}}$$

$$= \frac{e^{(y-\hat{v})} - e^{-(y-\hat{v})}}{e^{+(y-\hat{v})} + e^{-(y-\hat{v})}} = \tanh(y-\hat{v})$$

1.11. BPSK signal

$$y(i) = x + v(i)$$

Similar to previous problem,

$$E(x \mid \underline{y} = \underline{y}) = \frac{f(\underline{y} \mid x=1) - f(\underline{y} \mid x=-1)}{f(\underline{y} \mid x=1) + f(\underline{y} \mid x=-1)}$$

$$f(\underline{y} \mid x=1) = N(\underline{1}, \sigma_v^2 I), \quad f(\underline{y} \mid x=-1) = N(\underline{-1}, \sigma_v^2 I)$$

$$\hat{x}_N = \frac{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}(y_n-1)^2} - e^{-1/2\sigma_v^2 \sum_{n=0}^{N-1}(y_n+1)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(y_n-1)^2} + e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(y_n+1)^2}}$$

$$= \frac{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1} -2y_n} - e^{-1/2\sigma_v^2 \sum_{n=0}^{N-1} 2y_n}}{e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1} -2y_n} + e^{-1/2\sigma^2 \sum_{n=0}^{N-1} 2y_n}}$$

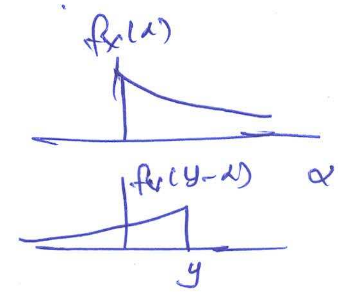
$$= \frac{e^{+\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n} - e^{-1/\sigma_v^2 \sum_{n=0}^{N-1} y_n}}{e^{1/\sigma^2 \sum_{n=0}^{N-1} y_n} + e^{-1/\sigma^2 \sum_{n=0}^{N-1} y_n}}$$

$$\hat{x}_N = \tanh\left(\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} y_n\right)$$

1.13 – Exponential distribution

$$y = x + v$$

$$f(x) = \lambda_1 e^{-\lambda_1 x} \qquad f(v) = \lambda_2 e^{-\lambda_2 v}.$$



a) To find pdf of y:

$$f(y) = \int_0^y f_x(\alpha) f_v(y-\alpha)\, d\alpha$$

$$= \lambda_1\lambda_2 \int_0^y e^{-(\lambda_1-\lambda_2)\alpha} \cdot e^{-\lambda_2 y}\, d\alpha$$

$$= \lambda_1\lambda_2 e^{-\lambda_2 y} \int_0^y e^{-(\lambda_1-\lambda_2)\alpha}\, d\alpha$$

$$\boxed{f(y) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 y}\left[e^{-(\lambda_1-\lambda_2)y} - 1\right]}$$

b) To find joint pdf of x and y:

$$y = x + V$$

$$f(y|x) = f_V(y-x) = \lambda_2 e^{-\lambda_2(y-x)}$$

$$f(x,y) = f(x) \cdot f(y|x)$$

$$= \lambda_1\lambda_2 e^{-\lambda_1 x} \cdot e^{-\lambda_2 y} \cdot e^{\lambda_2 x}$$

$$f(x,y) = \lambda_1\lambda_2 e^{-(\lambda_1-\lambda_2)x} \cdot e^{-\lambda_2 y} \qquad x \geq 0,\ y \geq 0$$

$$x \leq y$$

c) <u>Least mean square estimate of x given y</u>:

$$\hat{x} = E[X|Y=y]$$

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{\lambda_1\lambda_2 e^{-(\lambda_1-\lambda_2)x} \cdot e^{-\lambda_2 y}}{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} \cdot e^{-\lambda_2 y} \cdot \left[e^{-(\lambda_1-\lambda_2)y} - 1\right]}$$

$$= (\lambda_2-\lambda_1) \frac{e^{-(\lambda_1-\lambda_2)x}}{\left[e^{-(\lambda_1-\lambda_2)y} - 1\right]}$$

$$x \geq 0, \quad y \geq x$$

$$\hat{x} = \int_0^y x \cdot f(x|y)\,dx$$

$$= \frac{\lambda_2-\lambda_1}{[e^{(\lambda_2-\lambda_1)y}-1]} \int_0^y x\, e^{(\lambda_2-\lambda_1)x}\,dx$$

Integrating by parts we get,

$$= \frac{\lambda_2-\lambda_1}{[e^{(\lambda_2-\lambda_1)y}-1]} \left[\frac{y\, e^{(\lambda_2-\lambda_1)y}}{\lambda_2-\lambda_1} - \frac{e^{(\lambda_2-\lambda_1)y}}{(\lambda_2-\lambda_1)^2} + \frac{1}{(\lambda_2-\lambda_1)^2} \right]$$

Simplifying we get

$$\cancel{\hat{x} = \frac{1}{\lambda_1-\lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-(\lambda_1+\lambda_2)}}}$$

$$\boxed{\hat{x} = \frac{1}{\lambda_1-\lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-\lambda_2 y} - e^{-\lambda_1 y}} \cdot y}$$

[redacted]

2.21 - Estimation of x^2

$$y = x + v$$

We want to find the estimate of x^2 from $\{y, y^2\}$.

$$y^2 = x^2 + 2xv + v^2.$$

The parameter to be estimated has non-zero mean.

$\therefore$ The estimator is given by:

$$\hat{x}^2 = E[x^2] - K_0\,[y - E(y^2)]$$

K_0 is the solution for:

$$K_0 R_{(y^2)} = R_{(x^2 y^2)}$$

$$R_{y^2} = E[y^4] - [E(y^2)]^2$$

$$E[y^4] = E[(x^2+v^2+2xv)(x^2+v^2+2xv)]$$

$$R_{y^2} = E[x^4 + x^2v^2 + 2x^3v + v^2x^2 + v^4 + 2xv^3 +$$

$$2x^3v + 2xv^3 + 4x^2v^2] - (\sigma_x^2+\sigma_v^2)^2$$

$$= 3(\sigma_x^2+\sigma_v^2)^2 - (\sigma_x^2+\sigma_v^2)^2 = 2(\sigma_x^2+\sigma_v^2)^2$$

$$R_{x^2y^2} = E[(x^2-\sigma_x^2)(y^2-(\sigma_x^2+\sigma_v^2)^2)]$$

$$= E[x^2y^2 - \sigma_x^2 y^2 - (\sigma_x^2+\sigma_v^2)x^2 + \sigma_x^2(\sigma_x^2+\sigma_v^2)^2]$$

$$E[x^2y^2] = E[x^4 + x^2v^2 + 2x^3v]$$

$$= 3\sigma_x^4 + \sigma_x^2\sigma_v^2.$$

$$R_{x^2y^2} = 3\sigma_x^4 + \sigma_x^2\sigma_v^2 - (\sigma_x^2+\sigma_v^2)\sigma_x^2 + \sigma_x^2(\sigma_x^2+\sigma_v^2)$$

$$= 2\sigma_x^4.$$

$$\therefore K_0 = R_{x^2y^2} R_{y^2}^{-1}$$

$$= \frac{2\sigma_x^4}{2(\sigma_x^2+\sigma_v^2)^2}$$

$\therefore$ The estimator of x^2 is.

$$\boxed{\hat{x}^2 = \sigma_x^2 + \frac{\sigma_x^4}{(\sigma_x^2+\sigma_v^2)^2}(y^2 - \sigma_x^2 - \sigma_v^2)}$$

2.23 - Useful identity

x & y are dependent real zero valued random variables.

$$\hat{x} = K_1 y \quad : \quad K_1 R_y = R_{xy}$$

$$\hat{y} = K_2 x \quad : \quad K_2 R_x = R_{xy} \quad [R_{xy} = R_{yx}]$$

Error of estimation. $\tilde{x} = x - \hat{x}$, $\tilde{y} = y - \hat{y}$

$$R_{\tilde{x}} = E\left[(x-\hat{x})(x-\hat{x})^T\right]$$

$$= E\left[(x-\hat{x})x^T\right] = R_x - K_1 R_{xy}$$

$$R_{\tilde{y}} = R_y - K_2 R_{xy}$$

$$R_{\tilde{x}}^{-1} = R_x^{-1} - R_{xy} K_1^{-1}$$

$$= \cancel{R_x^{-1}} - R_{xy} \cancel{R_y} R_{xy}^{-1}$$

$$R_x R_{\tilde{x}}^{-1} = I - R_x R_{xy}^{-1} K_1^{-1}$$

$$R_x R_{\tilde{x}}^{-1} \hat{x} = K_1 y - R_x R_{xy}^{-1} K_1^{-1} K_1 y$$

$$= (K_1 - R_x R_{xy}^{-1}) y$$

$$= (R_{xy} R_y^{-1} - R_x R_{xy}^{-1}) y$$

$$= R_{xy} (R_y^{-1} - R_{xy}^{-1} R_x R_{xy}^{-1}) y$$

$$\boxed{R_x R_{\tilde{x}}^{-1} \hat{x} = R_{xy} R_{\tilde{y}}^{-1} y}$$

$$\because R_{\tilde{y}}^{-1} = R_y^{-1} - R_{xy}^{-1} K_2^{-1}$$

$$= R_y^{-1} - R_{xy}^{-1} R_x R_{xy}^{-1}$$

$K_1 R_y = R_{xy}$

$K_1 = R_{xy} R_y^{-1}$

$K_2 R_x = R_{xy}$

$K_2 = R_{xy} R_a^{-1}$

1.13 b

$$E[x \mid \underline{y}=\underline{y}] = \frac{P \cdot f(\underline{y} \mid x=1) - (1-P) f_y(\underline{y} \mid x=-1)}{P f(\underline{y} \mid x=1) + (1-P) f(\underline{y} \mid x=-1)}$$

$$= \frac{P \cdot e^{\left[\frac{1}{\sigma_v^2} \sum_{n=0}^{N-1} y_n\right]} - (1-P)\, e^{-1/\sigma_v^2 \sum_{n=0}^{N-1} y_n}}{P \cdot e^{\frac{1}{\sigma_v^2} \sum_{n=0}^{N-1} y_n} + (1-P)\, e^{-1/\sigma_v^2 \sum_{n=0}^{N-1} y_n}}$$

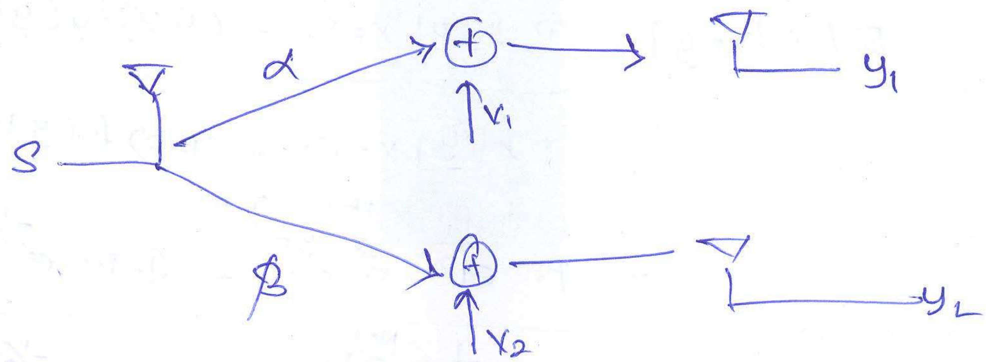
$$= \frac{e^{\ln P} \cdot e^{(\cdot)} - e^{\ln(1-P)} \cdot e^{-(\,)}}{e^{\ln P} \cdot e^{(\,)} + e^{\ln(1-P)} \cdot e^{-(\,)}}$$

$$= \frac{e^{\frac{\ln P}{2}} \cdot e^{\frac{\ln(1-P)}{2}}}{e^{\frac{\ln P}{2}} \cdot e^{\frac{\ln(1-P)}{2}}} \left[\frac{e^{\frac{\ln P - \ln(1-P)}{2}} \cdot e^{(\cdot)} - e^{\frac{-\ln(P) - \ln(1-P)}{2}} \cdot e^{(\cdot)}}{(\quad) + (\quad)}\right]$$

$$= \frac{e^{\left[\frac{1}{2} \ln \frac{P}{1-P} + \frac{1}{\sigma_v^2} \sum_{n=0}^{N-1} y_n\right]} - e^{-[\quad]}}{(\quad) + (\quad)}$$

$$\hat{x} = \tanh\left[\frac{1}{2} \ln \frac{P}{1-P} + \sum_{n=0}^{N-1} \frac{y_n}{\sigma_v^2}\right]$$

II.25 Maximal Ratio combining



$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} s + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \underline{h}\, s + \underline{v}$$

Assume that the receiver process the observed vector

as: $r^H \underline{y}$: where $r = [a \;\; b]^T$.

$$r^H \underline{y} = [a \;\; b]^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} . s + a v_1 + b v_2.$$

$$= (a\alpha + b\beta) s + \underbrace{(a v_1 + b v_2)}_{\text{noise}}$$

Signal power to noise ratio after post processing,

$$SNR = \frac{|a\alpha + b\beta|^2 \sigma_s^2}{(|a|^2 + |b|^2)\sigma_v^2}$$

$$= \frac{\sigma_s^2 . |\underline{r}^H \underline{h}|^2}{\sigma_v^2 . \|r\|^2}$$

$|\underline{r}^H \underline{y}|$ is maximized only when the angle between $\underline{r}$ & $\underline{h}$ is equal to zero. i.e. $\underline{r} = \underline{h}$..

$$(SNR)_{max} = \frac{\sigma_s^2}{\sigma_v^2} \cdot (|\alpha|^2 + |\beta|^2)$$

optimal linear combination in order to maximize the SNR: $\hat{s} = \alpha^* y_1 + \beta^* y_2$.

II.26 MMSE linear Combining

In the same set up as in the previous problem, now we want to linearly combine y_1 & y_2 to get the minimum mean square error between s & $\hat{s}$.

$$\min_{a,b} E\left[s - [a \ b]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right]^2$$

$$\underline{y} = \underline{h}\, s + \underline{v} \qquad \underline{h} = [\alpha \ \beta]^T$$

MMSE linear estimator of s from $\underline{y}$ is given by

$$\hat{s} = (R_s^{-1} + h^H R_v^{-1} h)^{-1} h^H R_v^{-1} y$$

$$= \left[\frac{1}{\sigma_s^2} + \frac{1}{\sigma_v^2}(|\alpha|^2 + |\beta|^2)\right]^{-1} \cdot \frac{1}{\sigma_v^2} \cdot (\alpha^* y_1 + \beta^* y_2)$$

$$= \left[\frac{\sigma_v^2 + \sigma_s^2 \|h\|^2}{\sigma_s^2 \sigma_v^2}\right]^{-1} \frac{1}{\sigma_v^2} (\alpha^* y_1 + \beta^* y_2)$$

$$= \frac{\sigma_s^2 \sigma_v^2}{\sigma_v^2 + \sigma_s^2 \|h\|^2} \cdot \frac{1}{\sigma_v^2} (\alpha^* y_1 + \beta^* y_2)$$

$$\boxed{\hat{S} = \frac{1}{\frac{\sigma_v^2}{\sigma_s^2} + \|h\|^2} (\alpha^* y_1 + \beta^* y_2).}$$

$$MMSE = \left[R_s^{-1} + h^H R_v^{-1} h\right]^{-1}$$

$$= \frac{\sigma_v^2}{\frac{\sigma_v^2}{\sigma_s^2} + \|h\|^2}$$