EE 6110 - Adaptive Signal Processing

Quiz - I - Solutions

1)

1st trial - m; 2nd trial (n)

n \ m	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

$X = \min(m,n)$

n \ m	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

$\leftarrow Y=4$

$Y = \max(m,n)$

a) $$E(X) = 1\cdot\frac{7}{16} + 2\cdot\frac{5}{16} + 3\cdot\frac{3}{16} + 4\cdot\frac{1}{16}$$

$$= \frac{30}{16}$$

$$E(Y) = 1\cdot{}^1/_{16} + 2\cdot{}^3/_{16} + 3\cdot{}^5/_{16} + 4\cdot{}^7/_{16}$$

$$= \frac{50}{16}$$

b) optimal MMSE estimate of $X|Y=4$.

$$E[X|Y=4] = 16/7.$$

[Given $Y=4$, (m,n) has 7 possible combinations as marked in the table, each with prob. $1/7$.]

c) Linear MMSE estimator of X given Y:

$$K_0 R_y = R_{xy}.$$

$$R_y = E[Y^2] - [E(Y)]^2$$

$$= \frac{220}{256}.$$

$$\underline{B(XY)}: \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix} \leftarrow XY$$

each with probability $= 1/16$.

$$R_{xy} = B(XY) - B(X) \cdot B(Y)$$

$$= \frac{100}{16} - \frac{1500}{256} = \frac{100}{256}.$$

$$K_0 = \frac{\cancel{B}\, R(XY)}{R_y} = \frac{100}{256} \times \frac{256}{220}$$

$$= \frac{10}{22} = \frac{5}{11}.$$

$$\therefore \cancel{\hat{x} = \frac{5}{11} \cdot y}$$

$$\hat{x} = \bar{x} + \frac{\sigma_{xy}}{\sigma_y^2}(y - \bar{y})$$

$$= \frac{30}{16} + \frac{5}{11}\left(y - \frac{50}{16}\right)$$

$$\hat{x} = \frac{5}{11}(y+1)$$

2. $y_1 = 2x + v_1$

$y_2 = x + v_2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y \quad = A x + v.$$

a) LMMSE of $x | y_1$:

$R_{y_1} = 5.$

$R_{xy_1} = E[2x^2 + xv_1] = 2.$

$\therefore \hat{x} = \frac{2}{5} y_1.$

b) optimal MMSE of $x | y_1 y_2$:

$K_o R_y = R_{xy}$

$$R_y = E\begin{bmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

$$E[x y_1 \quad x y_2] = [2 \quad 1] = R_{xy}$$

$$K_o = R_{xy} R_y^{-1} = [2 \quad 1] \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{6} [2 \quad 1] \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$$= \tfrac{1}{6} [2 \quad 1]$$

$$\boxed{\hat{X} = \frac{1}{6}\left[2Y_1 + Y_2\right]}$$

c) $E[\hat{X}] = \frac{1}{2}\left[E(Y_1) + E(Y_2)\right] = 0 = E(X).$

$\therefore$ The estimator is unbiased.

$$MSE = E\left[(X-\hat{X})^2\right] = E[X^2] + E[\hat{X}^2] - 2.E[X\hat{X}]$$

$$= 1 + \frac{11}{4} - 3$$

$$= \frac{3}{4}$$