Note Title 30-07-2014

EE 6110 Adaptive Signal Processing

Prerequisites: Probability Theory (Necessary)
Estimation Theory (Optional)

Textbook Fundamentals of Adaptive filtering
by Ali Sayed Wiley Indian Edition

Syllabus

- Optimal Estimation
- Linear Estimation
- Steepest Descent Algorithms
- Stochastic Gradient Algorithms → adaptive filters.
- Performance of adaptive filters.

Evaluation

Assignments	10%
Quiz 1 & 2	20+20
Final Exam	50%

Review of Probability Basics

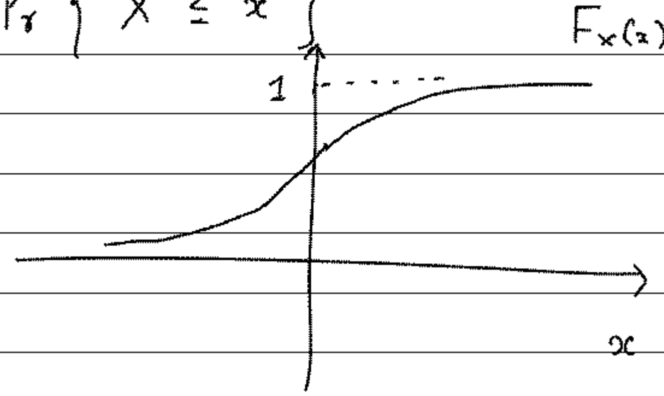
X is a Random Variable.

Cumulative distribution function of X (cdf)

$$F_X(x) = P_r\{X \leq x\}$$

$\uparrow$

increasing function of x.



$$F_X(-\infty) = 0$$

$$F_X(\infty) = 1$$

Note $\quad P_r\{a < X \leq b\} = F_X(b) - F_X(a)$

Probability density function (pdf) of X

$$f_X(x) = \frac{d}{dx} F_X(x) \quad \text{(if derivative exists)}$$

$$= \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$$

$$P_r\{a < X \leq b\} = \int_a^b f_X(x)\, dx$$

$$f_X(x) \geq 0 \quad \text{for all } x.$$

Conditional distribution

Let X be random variable

M be an event (eg. $M = \{a < X \leq b\}$)

Conditional distribution of X given M

$$F_X(x|M) = \Pr\{X \leq x | M\}$$

conditional probability

$$= \frac{\Pr\{X \leq x, M\}}{\Pr\{M\}}$$

Conditional density of X given M

$$f_X(x|M) = \frac{d}{dx} F_X(x|M)$$

Example:

X be RV denoting life time of a machine.

Let $M = \{X > t_0\}$

We know $F_X(x)$, $f_X(x)$

Want to find $F_X(x/M)$, $f_X(x|M)$

$$F_X(x|M) = Pr\{X \le x | M\}$$

$$= \frac{Pr\{X \le x,\ X > t_0\}}{Pr\{X > t_0\}}$$

$$F_X(x|M) = \begin{cases} 0 & ; \quad x \le t_0 \\ \dfrac{F_X(x) - F_X(t_0)}{1 - F_X(t_0)} & ; \quad x > t_0 \end{cases}$$

Conditional pdf

$$f_X(x|M) = \begin{cases} 0 & ; \quad x \le t_0 \\ \dfrac{f_X(x)}{1 - F_X(t_0)} & ; \quad x > t_0 \end{cases}$$

Expected Values

X is random variable with pdf $f_X(x)$

Mean of $X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$ → if it exists

Mean Squared $= E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

≥ 0

Variance of X $= E(X^2) - \{E(X)\}^2 \geq 0$

$$= E\left[\left(X - E(X)\right)^2\right]$$

Suppose $g(\cdot)$ is any function.

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

Conditional Expectation.

M is an event

$$E(X|M) = \int_{-\infty}^{\infty} x\, f_X(x|M)\, dx$$

$$E(X^2|M) = \int_{-\infty}^{\infty} x^2 f_X(x|M)\, dx$$

Conditional Variance $= E(X^2/M) - \left(E(X/M)\right)^2$

Example: Let X be Gaussian with

pdf $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$E(X) = 0 \qquad Var(X) = 1$

$M = \{X > 0\}$

Need to Compute

$E(X|M) = ?$

$f_X(x|M)$

$Var(X|M) = ?$

$$f_X(x|M) = \begin{cases} 2 f_X(x) & ; x > 0 \\ 0 & ; x < 0 \end{cases}$$

$$E(X|M) = \int_0^{\infty} 2x\, f_X(x)\, dx = \sqrt{2/\pi}$$

$$E(X^2|M) = \int_0^{\infty} 2x^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = 1$$

$$Var(X|M) = 1 - 2/\pi$$

x ————————→

1/8 Joint Random Variables.

X and Y are two random Variables

Joint distribution function F_{XY}

$$F_{XY}(x,y) = P_r\{X \le x, Y \le y\}$$

Joint density function f_{XY}

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$ (if the derivative exists)

We also have.

$$F_{XY}(x,y) = \int_{\alpha=-\infty}^{x} \int_{\beta=-\infty}^{y} f_{XY}(\alpha,\beta)\, d\alpha\, d\beta$$

Marginal distribution function — Y can take any value from $-\infty$ to ∞

$$F_X(x) = P_r\{X \le x\}$$

$$= P_r\{X \le x, Y \le \infty\}$$

$$= F_{XY}(x, \infty)$$

Similarly $F_Y(y) = F_{XY}(\infty, y)$

Marginal pdf of X $$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx} F_{XY}(x, \infty)$$

$$= \frac{d}{dx} \int_{\alpha=-\infty}^{x} \int_{\beta=-\infty}^{\infty} f_{XY}(\alpha,\beta)\, d\alpha\, d\beta$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, \beta)\, d\beta$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(\alpha, y)\, d\alpha$$

x ———— x

Two random variables X and Y are independent if

$$F_{XY}(x,y) = F_X(x)\, F_Y(y)$$

for all x & y.

equivalently

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

x ———— x

Two random variables X and Y are called <u>uncorrelated</u> if

$$E(XY) = E(X)\, E(Y)$$

clearly from definitions

indepent $\Rightarrow$ uncorrelated.

uncorrelated $\not\Rightarrow$ independence. (except for jointly Gaussian random variables)

Property: If X and Y are uncorrelated then

$$\rightarrow \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Variance

Proof: $\mathrm{Var}(X+Y) = E\left((X+Y)^2\right) - \left\{E(X+Y)\right\}^2$

$E(X+Y) = E(X) + E(Y)$

$$= E\left(X^2 + Y^2 + 2XY\right) - \left(E(X) + E(Y)\right)^2$$

$$= E(X^2) + E(Y^2) + \cancel{2E(XY)} - \left(E(X)\right)^2 - \left(E(Y)\right)^2 - \cancel{2E(X)E(Y)}$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

* ——— *

Two random variables $(X \,\&\, Y)$ are called orthogonal

if $E(XY) = 0$

Note: If one of RV has zero mean then orthogonality imply uncorrelated

* ——— *

Example. Let Θ be RV uniformly distributed in $[-\pi$ to $\pi]$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } \theta \in [-\pi \text{ to } \pi] \\ 0 & \text{else} \end{cases}$$

Let $X = \cos\Theta$

$Y = \sin\Theta$

$X^2 + Y^2 = 1$

X and Y are not independent.

Are X and Y uncorrelated ?

$$E(X) = E(\cos\Theta) = \int_{-\pi}^{\pi} \cos\theta \cdot f_\Theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos\theta \frac{1}{2\pi}\, d\theta = 0$$

$$E(Y) = 0$$

$$E(XY) = \int_{-\pi}^{\pi} \cos\theta \cdot \sin\theta \frac{1}{2\pi}\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{\sin 2\theta}{4\pi}\, d\theta = 0$$

$$E(XY) = 0 = E(X)\,E(Y)$$

X and Y are uncorrelated (also orthogonal)

Conditional distribution / density

X & Y are two RV. $F_{XY}(x,y)$ joint cdf

$f_{XY}(x,y) \rightarrow$ joint pdf.

Conditional distribution of X given $Y \le y$

$$F_X(x \mid Y \le y) = \Pr\{X \le x \mid Y \le y\}$$

$$= \frac{\Pr\{X \le x,\ Y \le y\}}{\Pr\{Y \le y\}}$$

$$F_X(x \mid Y \le y) = \frac{F_{XY}(x,y)}{F_Y(y)}$$

Conditional density function of X given $Y = y$

$$f_X(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad (\text{if it exists})$$

Interpretation

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Pr\left(x < X < x+\Delta x \mid y < Y \le y+\Delta y\right)}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\dfrac{\Pr\{x < X \le x+\Delta x,\ y < Y \le y+\Delta y\}}{\Delta x\ \Delta y}}{\dfrac{\Pr\{y < Y \le y+\Delta y\}}{\Delta y}}$$

$$= \frac{f_{XY}(x,y)}{f_Y(y)} \rightarrow \text{Conditional density of } X \text{ given } Y = y$$

$$= f_X(x|y)$$

Conditional Expectation of X given $Y = y$.

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x\, f_X(x|y)\, dx$$

↓

will be a function of y.

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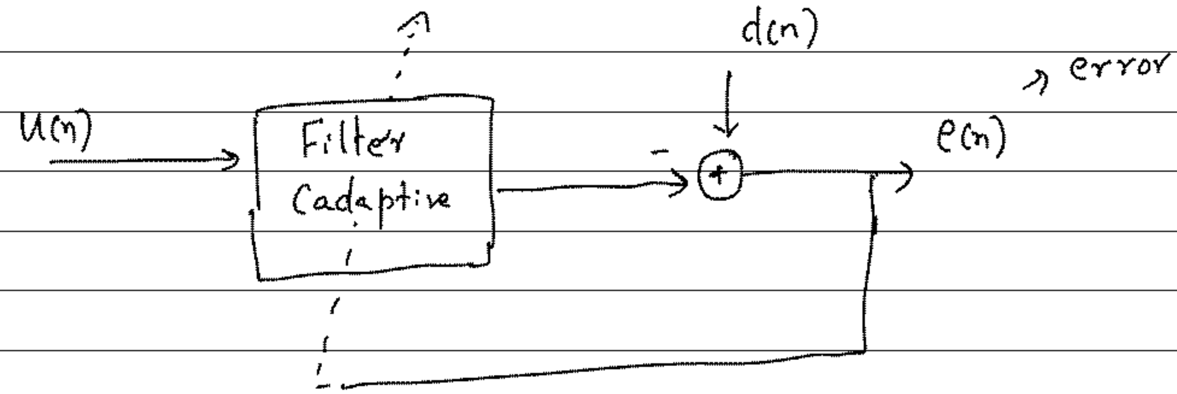
Preview of Adaptive Filters.

Typical Problem: $n \to$ time index

input signal $u(n)$ } sequence of

desired signal $d(n)$ } random variables



Goal: Minimize $E|e(n)|^2$

Adapt the filter based on $e(n)$

- If joint statistics of $u(n)$ and $d(n)$ are known then optimal filter can be computed in a single step

- Suppose joint statistics of $u(n)$ and $d(n)$ are not known

  or

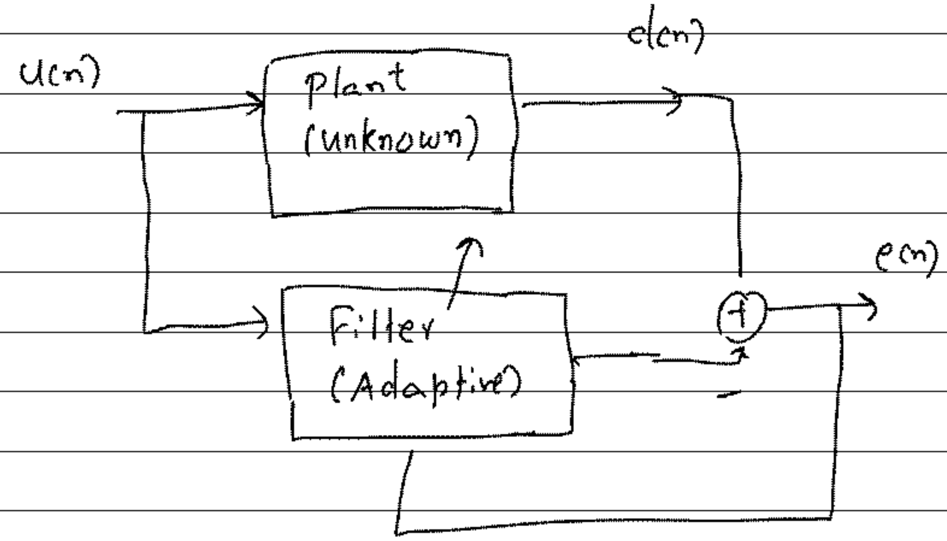
  Suppose statistics change with time

} $\Rightarrow$ Need for adaptive filtering

Question: how do we learn statistics over time and will the adaptive filter converge to optimal filter?

Adaptive Filtering Applications

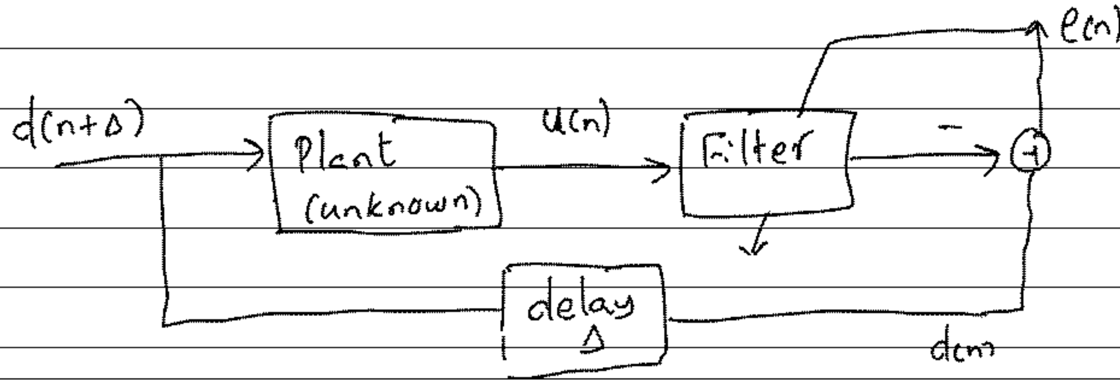
(a) System Identification



When $|e(n)|^2$ is small, adaptive filter response is close to plant response.

e.g: Channel estimation problem in communication

(b) Equalization problem



Here in the steady state when $e^2(n)$ is small, filter response is close to inverse of plant response

(c) other applications include echo cancellation, interference cancellation

x ———————— x