Assignment - II - Solutions

III. 3) optimal step size

$$\mu_{opt}(n) = \frac{\tilde{\underline{w}}^H(n) R_y^2 \tilde{\underline{w}}(n)}{\tilde{\underline{w}}^*(n) R_y^3 \tilde{\underline{w}}(n)}.$$

we know that $\nabla_w J(\underline{w}(n)) = \underline{w}^*(n) R_y - R_{xy}$

$$= \underline{w}^*(n) R_y - \underline{w}_{opt}^* R_y$$

$$= (\underline{w}(n) - \underline{w}_{opt})^* R_y$$

$$= \tilde{w}(n)^* R_y \rightarrow \text{Row vector.}$$

$$\frac{\| \nabla_w J(\underline{w}(n)\|^2}{(\nabla_w J(\underline{w}(n)) R_y [\nabla_w J(\underline{w}(n))]} = \frac{\tilde{w}^*(n) R_y . R_y \underline{w}(n)}{\tilde{\underline{w}}^*(n) R_y . R_y . R_y . \tilde{\underline{w}}(n)}$$

$$= \mu_{opt}(n).$$

4)

$$\mu^{opt}(n) = \frac{\tilde{w}^*(n) R_y^2 \tilde{w}(n)}{\tilde{w}^*(n) R_y^3 \hat{w}(n)}$$

$$= \frac{\tilde{w}^*(n) U \Lambda^2 U^H \tilde{w}(n)}{\tilde{w}^*(n) U \Lambda^3 U^H \tilde{w}(n)} \quad \because R_y = U \Lambda U^H$$

$$= \frac{(U\tilde{w}(n))^* \Lambda^2 [U \tilde{w}(n))}{[U\tilde{w}(n)]^* \Lambda^3 [U \tilde{w}(n)]}$$

$$= \frac{\underline{x}_n^{*} \underline{x}_n}{\underline{x}_n^{*} \Lambda \underline{x}_n} \qquad \Lambda = \mathrm{diag}(\lambda_1 \ldots \lambda_N)$$

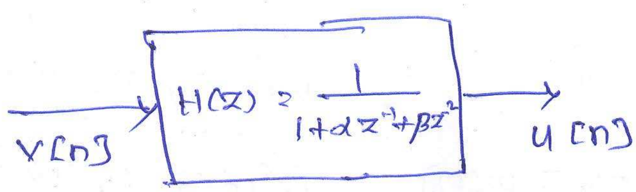
$$= \frac{\sum |x_i|^2}{\sum |\lambda_i x_i|^2}$$

$$\therefore \quad 1/\lambda_{max} \le \|w(i)\| \le 1/\lambda_{min}$$

8) <u>Prediction Problem:</u>

$$u[n] + \alpha u[n-1] + \beta u[n-2] = v[n] \quad \text{——} (*)$$

↓ zero mean white seq.



This is second order auto regressive model.

multiplying (*) by $\cancel{u^*[n+k]}$ $u[n-k]$ and taking expectation, we get the following "Yule-Walker" equation for the above AR model.

$$\cancel{E[u[n].u^*[n+k]] + \alpha . E[u^*[n+k].u[n-1]]}$$
$$\cancel{+ \beta\, E[u[n-2]\, u^*[n+k]] = E[v[n]\, u^*[n+k]]}$$

$$\cancel{R_u[k] + R_u[k+1]}$$

$$E[u(n)u^*(n-k)] + \alpha . E[u(n)u\, u(n-1)u^*(n-k)]$$

$$+ \beta . E[u(n-2)u^*(n-k)] = E[v(n).u^*(n-k)]$$

$$R_u(k) + \alpha . R_u(k-1) + \beta . R_u(k-2) = \begin{cases} E[v^2(n)], & \text{if } k=0 \\ 0 & \text{else.} \end{cases}$$

Substituting $k=0,1,2$ in the above eqn.

$$R_u(0) + \alpha . R_u(-1) + \beta R_u(-2) = \sigma_v^2$$

$$R_u(1) + \alpha . R_u(0) + \beta R_u(-1) = 0$$

$$R_u(2) + \alpha . R_u(1) + \beta . R_u(0) = 0$$

Since the process is stationary,

$$\underbrace{\begin{bmatrix} 1 & \alpha & \beta \\ \alpha & 1+\beta & 0 \\ \beta & \alpha & 1 \end{bmatrix}}_{A} \begin{bmatrix} R_u(0) \\ R_u(1) \\ R_u(2) \end{bmatrix} = \begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix}$$

a) Roots of the characteristic eqn.

$$z^2 + \underbrace{\alpha z}_{\text{sum of roots}} + \overbrace{\beta}^{\text{product of roots.}} = 0$$

Stable $\rightarrow$ $|z_1| < 1$, $|z_2| < 1$ $\Rightarrow$ $|z_1 z_2| < 1$, $|\beta| < 1$.

$\underbrace{z_1 = z_2^*}_{\text{since } \alpha, \beta \text{ are real.}}$ $\Rightarrow$ $|z_1|^2 = \beta$, $|z_1 + z_2| < 2\sqrt{\beta} \leq 1+\beta$

$|\alpha| \leq 1+\beta$

b) $R_u = E[u^* u]$

$$= E\left\{ \begin{bmatrix} u(i-1) \\ u(i-2) \end{bmatrix} \begin{bmatrix} u^*(i-1) & u^*(i-2) \end{bmatrix} \right\}$$

$$= \begin{bmatrix} R_u(0) & R_u(1) \\ R_u^*(1) & R_u(0) \end{bmatrix}$$

$$R_{du} = E[du^*] = E\left\{ [u(i)] \begin{bmatrix} u^*(i-1) \\ u^*(i-2) \end{bmatrix} \right\} = \begin{bmatrix} R_u(1) \\ R_u(2) \end{bmatrix}$$

$$\begin{bmatrix} R_u(0) \\ R_u(1) \\ R_u(2) \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \; R_u = \frac{\sigma_v^2}{\text{Det } A} \begin{bmatrix} 1+\beta & -\alpha \\ -\alpha & 1+\beta \end{bmatrix}, \quad R_{du} = \frac{\sigma_v^2}{\text{Det } A} \begin{bmatrix} -\alpha \\ \alpha^2 - \beta^2 - \beta \end{bmatrix}$$

$$\text{Det } A = (1-\beta)\left[(1+\beta)^2 - \alpha^2\right] > 0.$$

c) Optimal weight vector.

$$w^o = R_{du}^* R_u = [-\alpha \;\; -\beta]$$

d) <u>Eigen value spread of Ru</u>

For a symmetric matrix: $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$$(a-\lambda_i)^2 - b^2 = 0$$

$$a - \lambda_i = \pm b$$

$$\lambda_i = a \pm b$$

$$\therefore \rho = \frac{a+|b|}{a-|b|}$$

$$\therefore \rho = \frac{\beta+1+|\alpha|}{\beta+1-|\alpha|}$$

e) <u>Optimal step size:</u>

$$\mu_{opt} = \frac{2}{\lambda_{max} + \lambda_{min}}$$

$$= \frac{(1-\beta)\left[(1+\beta)^2 - \alpha^2\right]}{(1+\beta) \cdot \sigma_v^2}$$

11) <u>Regularized Newton's method</u>

$$\underline{w}(n+1) = \underline{w}(n) - \mu \left[\epsilon I + \nabla^2_{\underline{w}} J(\underline{w}(n))\right]^{-1} \left[\nabla_{\underline{w}} J(\underline{w})\right]^*$$

$$\underline{w}(n+1) = \underline{w}(n) - \mu \left[\epsilon I + R_y\right]^{-1} \left[R_y \underline{w}(n) - R_{yx}\right]$$

$$= \underline{w}(n) - \mu \left[\epsilon I + R_y\right]^{-1} \left[R_y \underline{w}(n) - R_y \underline{w}_{opt}\right]$$

Subtracting w_{opt} on both sides,

$$\tilde{\underline{w}}(n+1) = \Big[I - \mu \underbrace{\left[\epsilon I + R_y\right]^{-1} R_y}_{A}\Big] \tilde{\underline{w}}(n)$$

Similar steps as given in lecture notes, leads to

$$|1 - \cancel{\mu \lambda_i}| < 1, \quad \forall i$$

$$|1 - \mu \gamma_i| < 1, \quad \gamma_i \text{ - eig. values of } A.$$

or $$0 < \mu < \frac{2}{\gamma_{max}}.$$

$$A = \left[\epsilon I + R_y\right]^{-1} R_y = Q \begin{bmatrix} \frac{1}{\epsilon + \lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\epsilon + \lambda_N} \end{bmatrix} Q^H Q \Lambda Q^H$$

$$= Q \cdot \begin{bmatrix} \frac{\lambda_1}{\epsilon + \lambda_1} & & \\ & \ddots & \\ & & \frac{\lambda_N}{\epsilon + \lambda_N} \end{bmatrix} Q^H.$$

$$\therefore \quad 0 < \mu < \frac{2}{\gamma_{max}} = \cancel{2(1+\epsilon)} \; 2 + \frac{2\epsilon}{\lambda_{max}}.$$

$$\text{optimal step size} = \frac{2}{\gamma_{max} + \gamma_{min}} = \frac{2}{\frac{\lambda_{max}}{\epsilon + \lambda_{max}} + \frac{\lambda_{min}}{\epsilon + \lambda_{min}}}.$$