EE 5150: Mathematical Methods for Signal Processing
Quiz I - Sep/06/2012

1. Say if the following statements are true or false and provide an explanation.

   (a) If vectors $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_3\}$ are linearly *dependent*, then there always exists scalars a, b such that $\underline{\boldsymbol{u}}_1 = a\underline{\boldsymbol{u}}_2 + b\underline{\boldsymbol{u}}_3$. [1 pt]

   (b) All the vectors connecting origin to the points on the line $y = 2x + 1$ is a subspace of $\mathbb{R}^2$. [1 pt]

   (c) If $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_3\}$ is a basis for vector space V then $\{\underline{\boldsymbol{u}}_1 + \underline{\boldsymbol{u}}_3, 3\underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_1 - \underline{\boldsymbol{u}}_3\}$ is also a basis for V. [2 pts]

   (d) If the matrix $\boldsymbol{A}$ of size $m \times n$ has rank m then (irrespective of n) solution always exists for the equation $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ for any vector $\underline{\boldsymbol{b}}$ of suitable size. [2 pts]

2. Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -7 & -3 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 8 & 6 \end{bmatrix}$.

   (a) Find a basis for column space of $\boldsymbol{A}$. [2 pts]

   (b) Find a basis for null space of $\boldsymbol{A}$. [2 pts]

   (c) If $\underline{\boldsymbol{b}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, check if solution exists for $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$. If so, find a solution. [2 pts]

3. (a) Let V be a vector space. Let $\{\underline{\boldsymbol{u}}_1, \cdots, \underline{\boldsymbol{u}}_m\}$ be some arbitrary vectors in V. Let $\{\alpha_{i,j} : i = 1 \text{ to } n \text{ and } j = 1 \text{ to } m\}$ be some arbitrary scalars. Consider the vectors $\underline{\boldsymbol{w}}_i$'s generated as

$$\underline{\boldsymbol{w}}_i = \sum_{j=1}^{m} \alpha_{i,j}\underline{\boldsymbol{u}}_j, \quad \text{for } i = 1 \text{ to } n$$

   Show that $\texttt{span}\{\underline{\boldsymbol{w}}_1, \cdots, \underline{\boldsymbol{w}}_n\}$ is a subspace of $\texttt{span}\{\underline{\boldsymbol{u}}_1, \cdots, \underline{\boldsymbol{u}}_m\}$. [3 pts]

   (b) For two matrices (with size given in subscript) $\boldsymbol{A}_{m\times n}$ and $\boldsymbol{B}_{n\times p}$, consider the product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$. Prove that $\texttt{rank}(\boldsymbol{C}) \leq \min\{\texttt{rank}(\boldsymbol{A}), \texttt{rank}(\boldsymbol{B})\}$. [3 pts]
   `Hint:` Result from part (a) may be useful.

   (c) Let $\underline{\boldsymbol{u}}$ be a non-zero $n \times 1$ vector in $\mathbb{R}^n$. Consider the $n \times n$ matrix $\boldsymbol{C} = \underline{\boldsymbol{u}}\ \underline{\boldsymbol{u}}^t$ where $(\cdot)^t$ denotes transpose operation. What is the rank of $\boldsymbol{C}$? [2 pts]
   `Hint:` Result from part (b) may be useful.