Final Exam: EE 5150 Math Methods (26 Nov)

Notations:

- $(\cdot)^T$ denotes transpose, $(\cdot)^*$ denotes conjugate transpose, $|\cdot|$ denotes absolute value
- $\boldsymbol{I}_n$ and $\mathbf{0}_n$ denote identity matrix and zero matrix of size $n \times n$ respectively
- $\boldsymbol{A}_{n\times m}$ denotes a matrix of size $n \times m$.

Useful identities:

- For square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
- $\det(\boldsymbol{I}_n + \boldsymbol{C}_{n\times m}\boldsymbol{D}_{m\times n}) = \det(\boldsymbol{I}_m + \boldsymbol{D}_{m\times n}\boldsymbol{C}_{n\times m})$, for any $\boldsymbol{C}$ and $\boldsymbol{D}$ of appropriate sizes

Remarks

- You are allowed to bring a single hand-written sheet of paper
- If anything is not clear, make/state your assumptions and proceed

Problems:

1. In $\mathbb{C}^3$ with standard inner product, let $\underline{\boldsymbol{x}} = \begin{bmatrix} 2 \\ 1+j \\ j \end{bmatrix}$ and $\underline{\boldsymbol{y}} = \begin{bmatrix} 2-j \\ 2 \\ 1+2j \end{bmatrix}$.

   (a) For the above vectors $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$, verify Cauchy-Schwarz inequality and Triangle inequality. [4 pts]

   (b) Find scalars α and β such that $\alpha\underline{\boldsymbol{x}} + \beta\underline{\boldsymbol{y}}$ and $\alpha\underline{\boldsymbol{x}} - \beta\underline{\boldsymbol{y}}$ are orthogonal. [2 pts]

2. Let $W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 4 \\ -3 \end{bmatrix}\}$ be a subspace of $\mathbb{R}^4$. Let $W^\perp$ be orthogonal complement of W, that is $W^\perp = \{\underline{\boldsymbol{x}} \in \mathbb{R}^4 \text{ such that } <\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}> = \underline{\boldsymbol{x}}^T\underline{\boldsymbol{y}} = 0, \ \forall\ \underline{\boldsymbol{y}} \in W\}$.

   (a) Find orthonormal basis for $W^\perp$. [3 pts]

   (b) For $\underline{\boldsymbol{v}} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, find projection of $\underline{\boldsymbol{v}}$ onto subspace W. [3 pts]

3. Find *all* the invariant subspaces in $\mathbb{R}^2$ for the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. [5 pts]

4. Give a matrix example for each of the following conditions.

   (a) 3×3 matrix which is not diagonalizable. [2 pts]

   (b) $\boldsymbol{A}_{2\times 2}$ such that $\boldsymbol{A}^2 - 5\boldsymbol{A} + 6\boldsymbol{I} = \boldsymbol{0}$. [2 pts]

   (c) $\boldsymbol{A}_{3\times 4}$ such that the *smallest* set containing linearly *dependent* columns of $\boldsymbol{A}$ has size 3. [2 pts]

   (d) $\boldsymbol{A}_{2\times 2}$ such that $\mathrm{col}(\boldsymbol{A}) = \mathrm{null}(\boldsymbol{A})$. [2 pts]

5. Let V be a finite dimensional real inner product space. Let $\underline{\boldsymbol{u}}$ be an unit vector in V, i.e., $< \underline{\boldsymbol{u}}, \underline{\boldsymbol{u}} >= 1$. Consider the *householder* transformation $H : V \to V$ defined as

$$H(\underline{\boldsymbol{x}}) \quad = \quad \underline{\boldsymbol{x}} - 2 < \underline{\boldsymbol{x}}, \underline{\boldsymbol{u}} > \underline{\boldsymbol{u}} \quad \text{for all } \underline{\boldsymbol{x}} \in V$$

   Show that

   (a) H is a linear transformation [2 pts]

   (b) $H^2 = I$, where I is the identity transformation [2 pts]

   (c) H is an isometry [2 pts]

6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of same size.

   (a) Show that $\mathrm{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \mathrm{rank}(\boldsymbol{A}) + \mathrm{rank}(\boldsymbol{B})$ [3 pts]

   (b) Show that $\mathrm{rank}(\boldsymbol{A} - \boldsymbol{B}) \geq |\mathrm{rank}(\boldsymbol{A}) - \mathrm{rank}(\boldsymbol{B})|$ [2 pts]

   `Hint:` Think of *sum* of subspaces

7. Consider a complex $n \times n$ matrix $\boldsymbol{A}$. Prove the following statements.

   (a) If λ is an eigen value of $\boldsymbol{A}$, then $|\lambda| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|$ where $a_{i,j}$ is the entry in i^{th} row, j^{th} column of $\boldsymbol{A}$ [2pts]

   (b) If λ is an eigen value of $\boldsymbol{A}^*\boldsymbol{A}$, then $\lambda \geq 0$. [2 pts]

   (c) If $\boldsymbol{A}$ is unitary and λ is an eigen value of $\boldsymbol{A}$, then $|\lambda| = 1$. [2 pts]

8. Let $\underline{\boldsymbol{u}} \in \mathbb{R}^n$ be such that $\underline{\boldsymbol{u}}^T \underline{\boldsymbol{u}} = 1$. Consider the matrix $\boldsymbol{A} = \boldsymbol{I}_n + \underline{\boldsymbol{u}}\underline{\boldsymbol{u}}^T$

   (a) Is $\boldsymbol{A}$ diagonalizable? Explain your answer. [2 pts]

   (b) Find all the eigen values of $\boldsymbol{A}$ and their algebraic multiplicity. [3 pts]

   (c) Find/describe all the eigen vectors of $\boldsymbol{A}$. [3 pts]

   `Hint:` Think about *geometry* of $\underline{\boldsymbol{u}\boldsymbol{u}}^T$