EE 5150: Applied Linear Algebra: Assignment 2

1. Check if the following subsets are subspaces of $\mathbb{R}^n$. Notation $\underline{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ with $x_i \in \mathbb{R}$.

   (a) $S = \{\underline{\boldsymbol{x}} : x_1 = 1\}$ (b) $S = \{\underline{\boldsymbol{x}} : x_2 \geq 0\}$
   (c) $S = \{\underline{\boldsymbol{x}} : x_1 x_2 = 0\}$ (d) $S = \{\underline{\boldsymbol{x}} : x_1 + 2x_2 - 5x_3 = 0\}$

2. Let vector space $\mathcal{V} = \mathbb{R}^3$. For the set of vectors U given below, answer the following

   i Check if $\mathtt{span}\{U\} = \mathcal{V}$.
   ii Find a basis (denote by $\tilde{U}$) for $\mathtt{span}\{U\}$.
   iii Extend the set $\tilde{U}$ (obtained in part [ii]) to form a basis for $\mathcal{V}$.

$$U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\} \qquad U = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} \right\}$$

$$U = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \right\} \qquad U = \left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix} \right\}$$

3. Consider the set $\mathcal{V} = \{\underline{\boldsymbol{x}} = (x_1, x_2),\ \text{with } x_2 > 0\}$. For $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ in $\mathcal{V}$ and $\alpha \in \mathbb{R}$, define the addition and scalar multiplication as

$$\begin{aligned} \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} &= (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 v_2) \\ \alpha \underline{\boldsymbol{u}} &= (\alpha u_1, u_2^{\alpha}) \end{aligned}$$

   Show that $\mathcal{V}$ is a vector space with operations defined above.

4. Consider a set of matrices of form $\mathcal{S} = \left\{ \begin{bmatrix} a & a+b \\ a-b & a+2b \end{bmatrix} : \quad a, b \in \mathbb{R} \right\}$.

   (a) Show that $\mathcal{S}$ is a vector space with usual addition and scalar multiplication.
   (b) Find two different basis for $\mathcal{S}$.

5. Answer if the following statements are true or false. Provide logical arguements (proof) to support your answer.

   (a) In a vector space, size of a spanning set can not be smaller than size of a linearly independent set.
   (b) For vector spaces V and W, if $\mathtt{dim}(V) = \mathtt{dim}(W)$ then $V = W$.
   (c) Suppose vectors from set $\mathcal{U} = \{\underline{\boldsymbol{u}}_1, \cdots, \underline{\boldsymbol{u}}_n\}$ can be written as linear combination of vectors from set $\mathcal{W} = \{\underline{\boldsymbol{w}}_1, \cdots, \underline{\boldsymbol{w}}_m\}$, that is, $\exists\ \alpha_{i,j}$ such that $\underline{\boldsymbol{u}}_i = \sum_{j=1}^{m} \alpha_{i,j} \underline{\boldsymbol{w}}_j$ for $i = 1, \cdots, n$. Then it follows that $\mathtt{span}(\mathcal{U}) = \mathtt{span}(\mathcal{W})$.