(2) $Y = H + W \rightarrow 1$ sent

$Y = W \rightarrow 0$ sent

$$E_s = \frac{1+0}{2} = \frac{1}{2}, \quad E_b = \frac{E_s}{\log_2 m} = \frac{1/2}{1} = \frac{1}{2}$$

$H \sim N(0,1)$

$W \sim N(0, \sigma^2)$ Independent, $\sigma^2 = \frac{N_0}{2}$

(a) $f(y|H_1) \sim N(0, 1+\sigma^2)$

$f(y|H_0) \sim N(0, \sigma^2)$

$$l(y) = \log\left(\frac{f(y|H_1)}{f(y|H_0)}\right) = \log\left(\frac{\frac{1}{\sqrt{2\pi(1+\sigma^2)}} e^{-\frac{1}{2}\frac{y^2}{1+\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}}\right)$$

$$= \log\left(\frac{\sigma}{\sqrt{1+\sigma^2}} e^{-\frac{y^2}{2}\left(\frac{1}{1+\sigma^2} - \frac{1}{\sigma^2}\right)}\right)$$

$$= \log\left(\frac{\sigma}{\sqrt{1+\sigma^2}}\right) + \cancel{\frac{y^2}{2}} \frac{y^2}{2}\left(\frac{1}{\sigma^2(1+\sigma^2)}\right)$$

$$l(y) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\frac{y^2}{2}\left(\frac{1}{\sigma^2(1+\sigma^2)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log\left(\frac{\sqrt{1+\sigma^2}}{\sigma}\right)$$

$$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2(1+\sigma^2)\log\left(\frac{\sqrt{1+\sigma^2}}{\sigma}\right)$$

$$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2(1+\sigma^2)\log\left(\sqrt{\frac{1}{\sigma^2}+1}\right)$$

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(b) $$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2(1+\sigma^2)\log\left(\sqrt{\frac{1}{\sigma^2}+1}\right)$$

$$\sigma^2 = \frac{N_0}{2}$$

$$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\frac{N_0}{2}\left(1+\frac{N_0}{2}\right)\frac{1}{2}\log\left(\frac{2}{N_0}+1\right)$$

$$y^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(2+N_0)N_0}{4}\log\left(1+\frac{2}{N_0}\right)$$

$$|y| \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sqrt{N_0(N_0+2)}}{2}\left(\log\left(1+\frac{2}{N_0}\right)\right)^{1/2}$$

~~Pe/o~~ =

Decide H_0 if

$$-\frac{\sqrt{N_0(N_0+2)}}{2}\left(\log\left(1+\frac{2}{N_0}\right)\right)^{1/2} < y < \frac{\sqrt{N_0(N_0+2)}}{2}\left(\log\left(1+\frac{2}{N_0}\right)\right)^{1/2}$$

$$\varepsilon_1 < y < \varepsilon_2$$

H_1 otherwise

$$P_{e|0} = 2\int_{\varepsilon_2}^{\infty} f(y|H_0)\,dy$$

$$= 2\int_{y=\frac{\sqrt{N_0(N_0+2)}}{2}\left(\log\left(1+\frac{2}{N_0}\right)\right)^{1/2}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\, e^{-\frac{1}{2}\frac{y^2}{N_0/2}}\, dy$$

Put $\frac{y}{\sqrt{N_0/2}} = x$, $\frac{dy}{\sqrt{N_0/2}} = dx$

$$\therefore P_{e|0} = 2\int_{x=\sqrt{\frac{N_o+2}{2}\log\left(1+\frac{2}{N_o}\right)}}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

$$= 2Q\left(\sqrt{\left(\frac{N_o+2}{2}\right)\log\left(1+\frac{2}{N_o}\right)}\right)$$

$E_b = \frac{1}{2}$, $\cancel{= 2Q\left(1+\frac{N_o}{\ }\right.}$

$2E_b = 1$

$\frac{2E_b}{N_o} = \frac{1}{2N_o}$

So, $\therefore N_o = \frac{N_o}{2E_b}$, $\frac{N_o}{2} = \frac{N_o}{4E_b}$

$$\therefore P_{e|0} = 2Q\left(\sqrt{\left(1+\frac{N_o}{4E_b}\right)\log\left(1+\frac{4E_b}{N_o}\right)}\right)$$

$$P_{e|1} = \int_{\varepsilon_1}^{\varepsilon_2} f(y|H_1)\,dy$$

$$= \frac{1}{\sqrt{2\pi\left(1+\frac{N_o}{2}\right)}}\int_{\varepsilon_1}^{\varepsilon_2} e^{-\frac{1}{2}\frac{y^2}{1+N_o/2}}\,dy$$

Put $\frac{y}{\sqrt{1+\frac{N_o}{2}}} = x$, $\frac{dy}{\sqrt{1+\frac{N_o}{2}}} = dx$

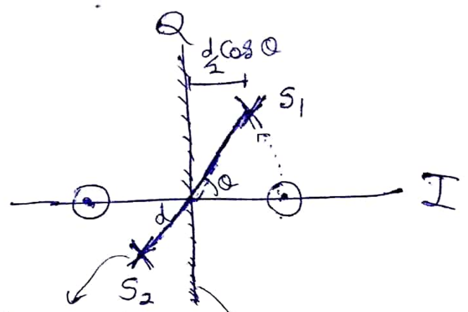
$$= \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{\frac{N_o}{2}\log\left(1+\frac{2}{N_o}\right)}}^{\sqrt{\frac{N_o}{2}\log\left(1+\frac{2}{N_o}\right)}} e^{-\frac{1}{2}x^2}\,dx$$

$$= 1 - 2Q\left(\sqrt{\frac{N_o}{2}\log\left(1+\frac{2}{N_o}\right)}\right)$$

$$= 1 - 2Q\left(\sqrt{\frac{N_o}{4E_b}\log\left(1+\frac{4E_b}{N_o}\right)}\right)$$

$$\therefore P_e = \frac{1}{2}\left[2Q\left(\sqrt{\left(1+\frac{N_o}{4E_b}\right)\log\left(1+\frac{4E_b}{N_o}\right)}\right) + 1 - 2Q\left(\sqrt{\frac{N_o}{4E_b}\log\left(1+\frac{4E_b}{N_o}\right)}\right)\right]$$

1 (a) $\theta < \frac{\pi}{2}$



noiseless signal points with carrier phase of 'θ'

mismatched decision boundary

(b) I channel noise ~~N~~ $n \sim N(0, \sigma^2)$

we get a boundary crossing when $n > \frac{d}{2}\cos\theta$

$$P_e = \frac{1}{2}\left(P_{e|s_1} + P_{e|s_2}\right)$$

By ~~symmetricity~~, symmetry, $P_{e|s_1} = P_{e|s_2}$

$$\therefore P_e = P_{e|s_1} = P\left(n > \frac{d}{2}\cos\theta\right)$$

$$= Q\left(\frac{\frac{d}{2}\cos\theta}{\sqrt{\sigma^2}}\right) \quad \text{... std 'Q' function form}$$

$$= Q\left(\frac{d\cos\theta}{2\sigma}\right)$$

Also, $E_b = d^2$, $N_o = \frac{\sigma^2}{2}$ $\quad \therefore P_e = Q\left(\sqrt{\frac{2E_b}{N_o}}\cos\theta\right)$

(c) $\theta \rightarrow \text{unif}\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$P(\theta) = \frac{2}{\pi} \quad \text{for } \theta = -\frac{\pi}{4} \text{ to } \frac{\pi}{4}$$

$$\cancel{P_e = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} Q\left[\sqrt{\frac{2E_b}{N_0}}\cos\theta\right] d\theta}$$

$$\overline{P_e} = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} Q\left[\sqrt{\frac{2E_b}{N_0}}\cos\theta\right] d\theta$$

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3 a) **Coherent Binary Signalling Scheme under AWGN**

The coherent Signalling Scheme for different constellations can be compared by power efficiency, $\eta_P = \frac{d^2}{E_b}$.

Signal set A

$E_b = \frac{E_s}{2}$; $\cancel{E_{s0}\ \int_0^{2} v^2(t)\,dt = 3/2}$.

$E_{s0} = \int_0^{2} v^2(t)\,dt = \frac{3}{2} = E_{s1} \Rightarrow E_s = \frac{3}{4} = E_b.$

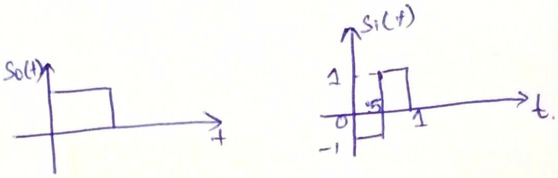
$\int v^2(t) = 2\cdot\int_0^{1}\left(\frac{3}{2}t\right)^2 dt = \frac{9}{2}\cdot\left[\frac{t^3}{3}\right]_0^1 = \frac{3}{2}$

$d = s_0(t) - s_1(t) = 2v(t) \Rightarrow d^2 = \int |s_0(t)-s_1(t)|^2\,dt$

$= 4\cdot\int_0^{2} v^2(t)\,dt = 4.E_b.$

$\Rightarrow (\eta_P)_A = \frac{4E_b}{E_b} = \underline{\underline{4}}.$

Signal set B.



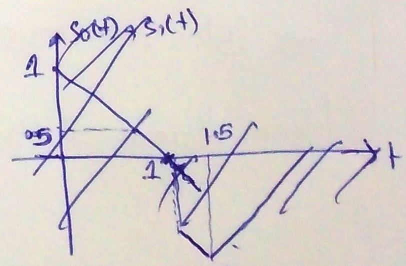
$E_b = E_s.$

$d = s_0(t) - s_1(t) = 2.I_{\{0,0.5\}}$

$d^2 = 4.\int_0^{0.5} 1\,dt = \underline{\underline{2}}.$

$E_b = 1 \Rightarrow [\eta_P]_B = 2.$

Signal set C.



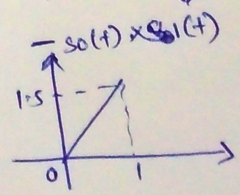
$E_{s0} = 1$

$E_{s1} = \int_0^{2} v^2(t)\,dt = \frac{3}{2}.$

$\Rightarrow E_s = E_b = \frac{\frac{3}{2}+1}{2} = \frac{5}{4}.$

$d = s_0(t) - s_1(t)$

$d^2 = \int |s_0(t)|^2 + |s_1(t)|^2 - 2\langle s_0(t), s_1(t)\rangle\,dt$



$= 1 + \frac{3}{2} - 2\cdot\int s_0(t)\,s_1(t)\,dt$

$= 1 + \frac{3}{2} + 2\left(\int_0^{1}\left(\frac{3}{2}t\right)dt\right)$

$= 1 + \frac{3}{2} + \frac{3}{2} = \underline{\underline{4}}.$

$\Rightarrow (\eta_P)_C = \frac{4}{5/4} = \frac{16}{5} = \underline{\underline{3.2}}$

Power efficiency is the least for Signal set B & hence Signal set B performs bad.

b). <u>Non coherent Signalling Scheme</u>

The performance for Noncoherent Signalling Scheme can be compared from the asymptotic error performance based on the correlation coefficient ρ if the signals are of equal energy.

Here, for Signal set C, $s_0(t)$ & $s_1(t)$ have different Energy. So, the performance cannot be evaluated ~~directly for Signal set C~~ using the expression for C.

$$P_e(\text{non coh}) \sim e^{-\frac{E_s}{2N_0}(1-|\rho|)}$$

∴, ~~Under~~ If the signals are of equal energy, the one with the minimum $|\rho|$ ~~be~~ performs better.

<u>Signal set A.</u>

$(\rho)_A = -1$

<u>Signal set B.</u>

$(\rho)_B = 0.$

$\Rightarrow |\rho|$ is minimum for B.

Hence Signal set (B) performs the best.

4) $y(n) = 2b(n) + b(n-1) + \omega(n)$

a) Given $\underline{b}$, $f(y(n) | \underline{b}) \sim f_W(y(n) - \mu(n))$ where $\mu(n) = 2b(n) + b(n-1)$ where $f_W(\cdot)$ is the pdf of $\omega(n)$. Since $\omega(n)$ is iid white, $y(n)/\underline{b}$ also will be independent.

b). $\underline{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}$, $\underline{b} = \begin{bmatrix} b(0) \\ b(1) \\ \vdots \\ b(N) \end{bmatrix}$

$$f(\underline{y}/\underline{b}) = \prod_{n=0}^{N} f(y(n)|\underline{b}) \quad (\because \text{by } 4(a), \{y(n)\} \text{ are independent}).$$

$$\left(\text{Also } y(n)|\underline{b} \sim f_W(y(n) - \mu(n)) = N(\mu(n), 1) \right)$$

$$= \prod_{n=0}^{N} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y(n)-\mu(n))^2}{2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^N e^{-\sum_{n=0}^{N} \frac{(y(n)-\mu(n))^2}{2}}$$

c) ML metric,

$$\hat{b} = \underset{\underline{b}}{\operatorname{argmax}}\ f(\underline{y}/\underline{b})$$

$$= \underset{\underline{b}}{\operatorname{argmin}}.\ \frac{\sum_{n=0}^{N}(y(n))^2 + \sum_{n=0}^{N}(\mu(n))^2 - 2 y(n)\mu(n).}{2}$$

$$= \underset{\underline{b}}{\operatorname{argmin}} \sum_{n=0}^{N}\left[(\mu(n))^2 - 2y(n)\mu(n)\right] \quad \left(\text{since } \sum y^2(n) \text{ is common in all terms}\right)$$

d). MLSE rule:

$$\hat{\underline{b}} = \arg\min_{\underline{b}} \sum_{n=0}^{N} \left[(u(n))^2 - 2y(n)u(n)\right]$$

$$= \arg\max_{\underline{b}} \sum_{n=0}^{N} u(n)\left[2y(n) - u(n)\right]$$

where $u(n) = 2b(n) + b(n-1)$

e) $N=4$; $b(0) = b(4) = 0$; $L=1 \Rightarrow M^L = 2^1 = \underline{\underline{2}}$

$y(0)=1$, $y(1)=2$, $y(2)=1$, $y(3)=-4$, $y(4)=-3$.

Branch metric, $\lambda_n = u(n)\left(2y(n) - u(n)\right) = \lambda_n(S_n \rightarrow S_{n+1})$

$\lambda_1(0\rightarrow 0) = u(1)\left(2y(1) - u(1)\right)$

$u(1) = 2b(n) + b(n-1) \;\Big|\; \begin{matrix} b(n)=0 \\ b(n-1)=0 \end{matrix}$

$= 0.$

$\lambda_2(0\rightarrow 0) = 0.$

$\lambda_3(0\rightarrow 0) = 0$

$\lambda_4(0\rightarrow 0) = 0$ $\left(\because \begin{matrix} b(4) \\ =0 \end{matrix}\right)$

$\lambda_1(0\rightarrow 1) = u(1)\left(2y(1) - u(1)\right)$

$u(1) = 2b(n) + b(n-1) \;\Big|\; \begin{matrix} b(n)=1 \\ b(n-1)=0 \end{matrix}$

$= 2.$

$\lambda_1(0\rightarrow 1) = 4.$

$\lambda_2(0\rightarrow 1) = 2(2\times 1 - 2) = 0$

$\lambda_3(0\rightarrow 1) = -20$

$\lambda_2(1\rightarrow 0) = u(1)\left(2y(2) - u(1)\right)$

$u(1) = 2b(n) + b(n-1) \;\Big|\; \begin{matrix} b(n-1)=1 \\ b(n)=0 \end{matrix}$

$= 1$

$= 2y(2) - 1 = \underline{\underline{1}}$

$\lambda_3(1\rightarrow 0) = 2y(3) - 1 = \underline{\underline{-9}}$

$\lambda_4(1\rightarrow 0) = \underline{\underline{-7}}$

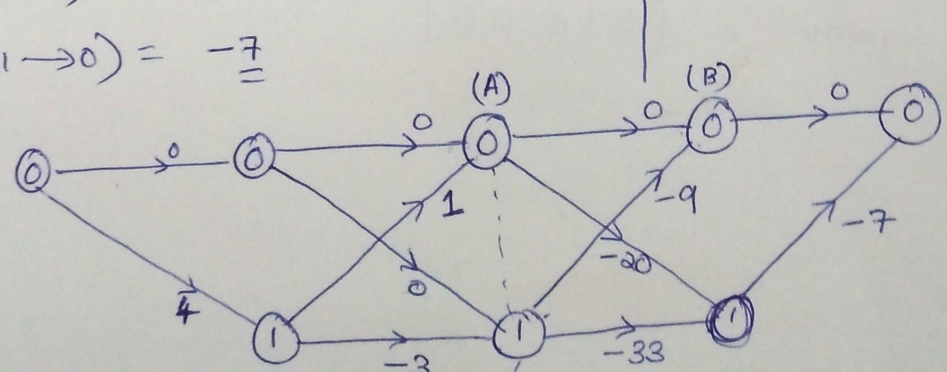
$\lambda_2(1\rightarrow 1) = 3(2y(2) - 3)$

$\left(u(1) = 2b(n) + b(n-1) \;\Big|\; \begin{matrix} b(n)=1 \\ b(n-1)=1 \end{matrix} \right)$

$= 3$

$= -3.$

$\lambda_3(1\rightarrow 1) = -33$



Vitterbi Algorithm

Survivor at state 0 at level 2,
(marked A in trellis)

$S_2(0)$

$\lambda_1(0\to0) + \lambda_2(0\to0) = 0$

$\lambda_1(0\to1) + \lambda_2(1\to0) = 5.$ $\Rightarrow$ survivor path $S_2(0) = 0 \to 0 \to 0.$

~~Accumulated metric, $\Lambda_2(0) = 0$, $\Lambda_2(1) = 13$~~

$S_2(1)$

$\lambda_1(0\to0) + \lambda_2(0\to1) = 0$

$\lambda_1(0\to1) + \lambda_2(1\to1) = 4 + -3 = 1$ $\Rightarrow$ $S_2(1) = 0\to1\to1$

Accumulated metric, $\Lambda_2(0) = 5.$

$\Lambda_2(1) = 1.$

$S_3(0)$

$\Lambda_2(0) + \lambda_3(0\to0) = 5$ $\Rightarrow$ $S_3(0) = 0\to1\to0\to0$

$\Lambda_2(1) + \lambda_3(1\to0) = -8$

$S_3(1)$

$\Lambda_2(0) + \lambda_3(0\to1) = -15$ $\Rightarrow$ $S_3(1) = 0\to1\to0\to1$

$\Lambda_2(1) + \lambda_3(1\to1) = -32.$

Accumulated metric, $\Lambda_3(0) = 5$

$\Lambda_3(1) = -15$

$S_4(0)$

$\Lambda_3(0) + \lambda_4(0\to0) = 5+0 = 5.$ $\Rightarrow S_4(0)$

$\Lambda_3(1) + \lambda_4(1\to0) = -15-7 = -22$

$\Rightarrow$ Survivor path at the end $= S_4(0) = 0\to1\to0\to0\to0$

$\therefore$ ML sequence $= [0\ 1\ 0\ 0\ 0]$