Signal Space Concepts

27 August 2017 22:53

Suppose we have a set of signals

$\{ s_1(t), s_2(t), \ldots, s_M(t) \}$

→ We want to develop vector representation for signals

→ (Replace waveforms with equivalent vectors

→ Geometric visualization of signals

Towards these we study

→ orthogonal Basis Expansion for Signals

→ Gram - Schmidt orthogonalization Procedure

→ Application to digital modulated waveforms

Some Definitions

Consider signal set

$$\mathcal{S} = \{ s_1(t), s_2(t), \ldots, s_M(t) \}$$

(finite energy)

$s_k(t)$ are complex valued waveforms

signal set

Signal Space defined by $\mathcal{S}$ is the set (collection) of all the waveforms (signals) of the form

$$\alpha_1 s_1(t) + \alpha_2 s_2(t) + \cdots + \alpha_M s_M(t)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_M$ are arbitrary. complex numbers

Consider the set

$$\mathcal{B} = \{ \psi_1(t), \ldots, \psi_p(t) \}$$

$\mathcal{B}$ is called orthonormal basis for signal space defined by signal set $\mathcal{S}$ if following two conditions are met

(i) $$\int_{-\infty}^{\infty} \psi_k(t)\, \psi_\ell^*(t)\, dt = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

$$1 \le k \le P$$
$$1 \le l \le P$$

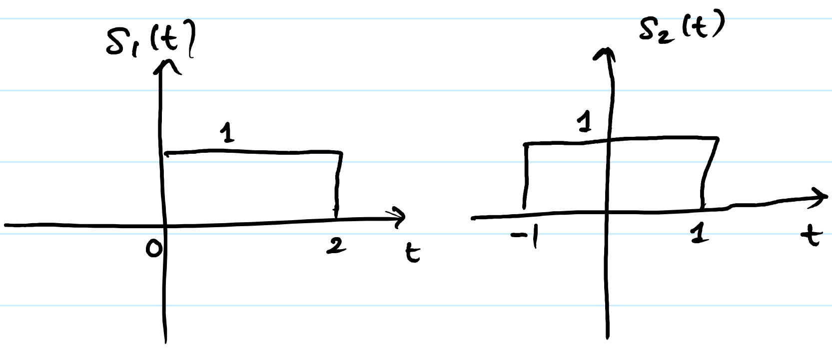
(ii) Any signal $s_i(t)$ in the signal set can be represented as

$$s_i(t) = \alpha_{i,1}\,\psi_1(t) + \alpha_{i,2}\,\psi_2(t) + \cdots + \alpha_{i,P}\,\psi_P(t)$$

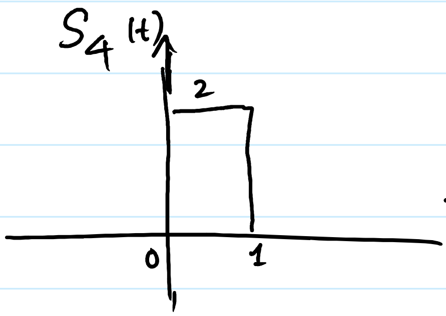
$$1 \le i \le M$$

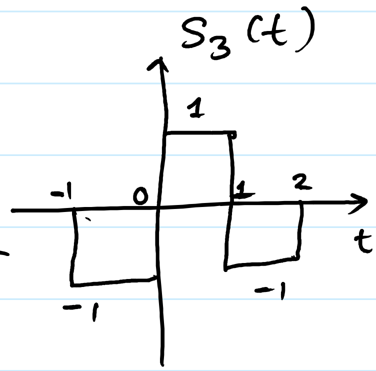
Using orthonormal basis, we get compact (vector) representations for our signals

Example: $M = 4$

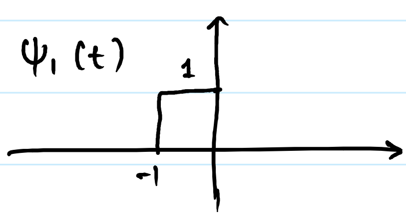


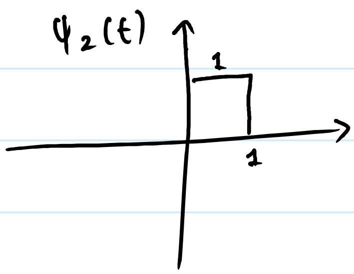
$S_4(t)$ $S_3(t)$

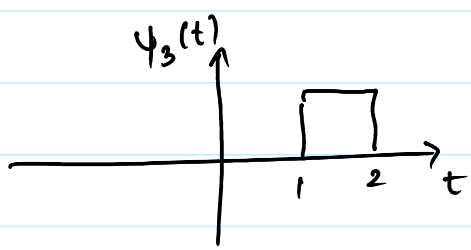




We can choose orthonormal basis as







$$s_1(t) = 0\,\psi_1(t) + 1\,\psi_2(t) + 1\cdot\psi_3(t)$$

Consider vector $\underline{s_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$s_2(t) = 1\cdot\psi_1(t) + 1\cdot\psi_2(t) + 0\,\psi_3(t)$$

$$\underline{s_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$s_4(t) = 0 \cdot \psi_1(t) + 2\,\psi_2(t) + 0 \cdot \psi_3(t)$

$$\underline{s_4} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$s_3(t) = (-1)\,\psi_1(t) + (1)\,\psi_2(t) + (-1)\,\psi_3(t)$

$$\underline{s_3} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$\underline{s_k}$ → obtained using coefficients of basis expansion of $s_k(t)$

→ Equivalent representation of signal $s_k(t)$

→ several operations involving $s_k(t)$ can be done equivalently using vector $\underline{s_k}$

x ———————————— x

Connections between signal waveforms and their vector representations

Consider Signal set (with finite energy)

$$\mathcal{S} = \{ s_1(t), s_2(t), \ldots, s_M(t) \}$$

and an orthonormal basis

for signal space

$$\mathcal{B} = \{ \psi_1(t), \psi_2(t), \ldots, \psi_p(t) \}$$

Note : $\boxed{P \leq M}$

P = number of basis functions
⇓
dimension of signal space

Consider basis expansion of $s_i(t)$

$$s_i(t) = a_{i,1}\, \psi_1(t) + \cdots + a_{i,P}\, \psi_P(t)$$

$a_{i,n} \rightarrow$ are complex numbers

Equivalent vector representation

$$\underline{s_i} = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,P} \end{bmatrix} \quad P \times 1 \text{ vector}$$

① Coefficient $a_{i,n}$ in basis expansion is got as follows

$$a_{i,n} = \langle s_i(t), \psi_n(t) \rangle$$

$$= \int_{-\infty}^{\infty} s_i(t)\, \psi_n^*(t)\, dt$$

Proof:

Recall $s_i(t) = \sum_{n=1}^{P} a_{i,q}\, \psi_q(t)$

Recall $s_i(t) = \sum_{q=1} a_{i,q} \psi_q(t)$

Consider $\langle s_i(t), \psi_n(t) \rangle$

$$= \int_{-\infty}^{\infty} s_i(t) \psi_n^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} \left(\sum_{q=1}^{P} a_{i,q} \psi_q(t) \right) \psi_n^*(t) \, dt$$

$$= \sum_{q=1}^{P} a_{i,q} \underbrace{\int_{-\infty}^{\infty} \psi_q(t) \psi_n^*(t) \, dt}_{= \begin{cases} 1 & \text{if } q = n \\ 0 & \text{else} \end{cases}}$$

$$= a_{i,n}$$

x ——————————— x

② Parseval's theorem

$\underline{s_i}$, Define norm of vector as

$$\begin{bmatrix} a_{i,1} \\ \vdots \\ a_{i,P} \end{bmatrix} \qquad \| \underline{s_i} \| = \sqrt{\sum_{q=1}^{P} |a_{i,q}|^2}$$

∞

$$\begin{bmatrix} a_{i,P} \end{bmatrix}_{P\times 1}$$

We have $\|\underline{s}_i\|^2 = \int_{-\infty}^{\infty} |s_i(t)|^2 dt$

$= \text{Energy of Signal } s_i(t)$

Proof: $\int_{-\infty}^{\infty} |s_i(t)|^2 dt = \int_{-\infty}^{\infty} s_i(t)\ s_i^*(t)\, dt$

$$= \int_{-\infty}^{\infty} \left(\sum_{q=1}^{P} a_{i,q}\ \psi_q(t)\right)\left(\sum_{l=1}^{P} a_{i,l}\ \psi_l(t)\right)^* dt$$

$$= \sum_{q=1}^{P} \sum_{l=1}^{P} a_{i,q}\ a_{i,l}^* \underbrace{\int_{-\infty}^{\infty} \psi_q(t)\ \psi_l^*(t)\, dt}_{= \begin{cases} 1 & \text{if } q = l \\ 0 & \text{else} \end{cases}}$$

$$= \sum_{q=1}^{P} |a_{i,q}|^2$$

$$= \|\underline{s}_i\|^2$$

③ Inner Product between Signals and inner product between vectors

$$s_k(t) = \sum_{q=1}^{P} a_{k,q} \, \psi_q(t)$$

$$s_i(t) = \sum_{q=1}^{P} a_{i,q} \, \psi_q(t)$$

$$\underline{s}_k = \begin{bmatrix} a_{k,1} \\ a_{k,2} \\ \vdots \\ a_{k,P} \end{bmatrix} \qquad \underline{s}_i = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,P} \end{bmatrix}$$

Inner product (defined as) $\langle \underline{s}_i, \underline{s}_k \rangle = \sum_{q=1}^{P} (a_{i,q})(a_{k,q}^{*})$

$$= \underbrace{\underline{s}_k^{*}}_{\text{conjugate \& transpose}} \underline{s}_i$$

We have

$$\langle s_i(t), s_k(t) \rangle = \langle \underline{s}_i, \underline{s}_k \rangle$$

that is $$\int_{-\infty}^{\infty} s_i(t)\, s_k^*(t)\, dt = \underline{s}_k \, \underline{s}_i$$

Prove this yourself.

Gram-Schmidt Orthogonalization Procedure

- A general procedure to obtain an orthonormal basis for any signal space
- step by step process

Let Signal set $S = \{ s_1(t), s_2(t), \ldots, s_M(t) \}$

Step 1:

Define $\phi_1(t) = s_1(t)$

If $\|\phi_1(t)\| \neq 0$ then

Choose $\psi_1(t) = \dfrac{\phi_1(t)}{\|\phi_1(t)\|}$

Note : $\|\phi_1(t)\|^2 = \int_{-\infty}^{\infty} |\phi_1(t)|^2 \, dt$

$= \text{Energy of } \phi_1(t)$

<u>Step K</u> ($k = 2$ to M)

Let $\{\psi_1(t), \psi_2(t), \ldots, \psi_\ell(t)\}$

denote set of basis functions obtained until <u>step $k-1$</u>

Now define

$$\phi_K(t) = s_k(t) - \sum_{q=1}^{\ell} \langle s_k(t), \psi_q(t) \rangle \psi_q(t)$$

If $\|\phi_k(t)\| \neq 0$ then

choose the next basis function

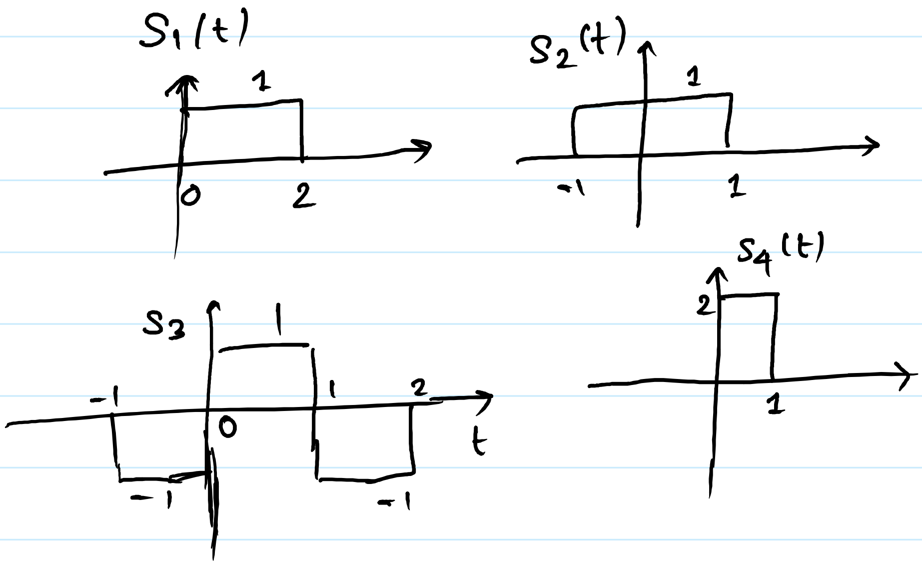
as $\psi_{\ell+1}(t) = \dfrac{\phi_k(t)}{}$

$\| \phi_k(t) \|$

If $\| \phi_k \| = 0$, then do not update basis set.

Proceed to next step $k+1$

Consider the signal set (same as before)

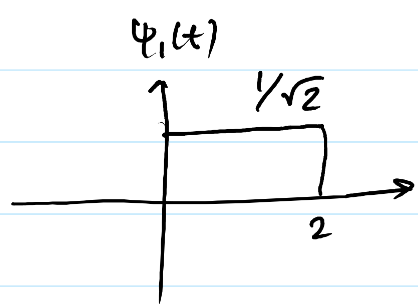


step 1 :

$$\phi_1(t) = s_1(t)$$

$$\| \phi_1(t) \| = \sqrt{2}$$

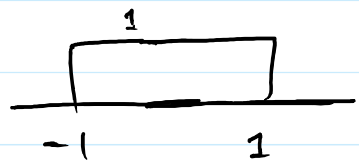
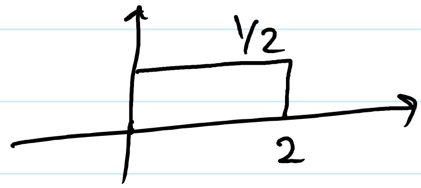
$$\psi_1(t) = \frac{s_1(t)}{\sqrt{2}}$$

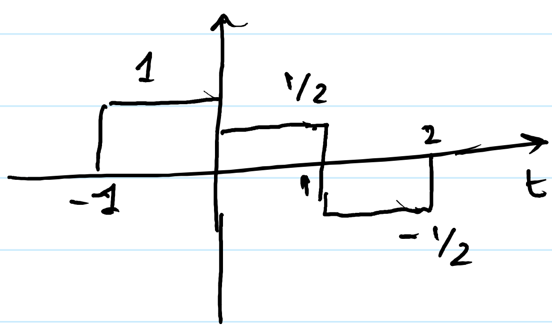


Step 2 :

$$\phi_2(t) = s_2(t) - \langle s_2(t), \psi_1(t) \rangle \, \psi_1(t)$$

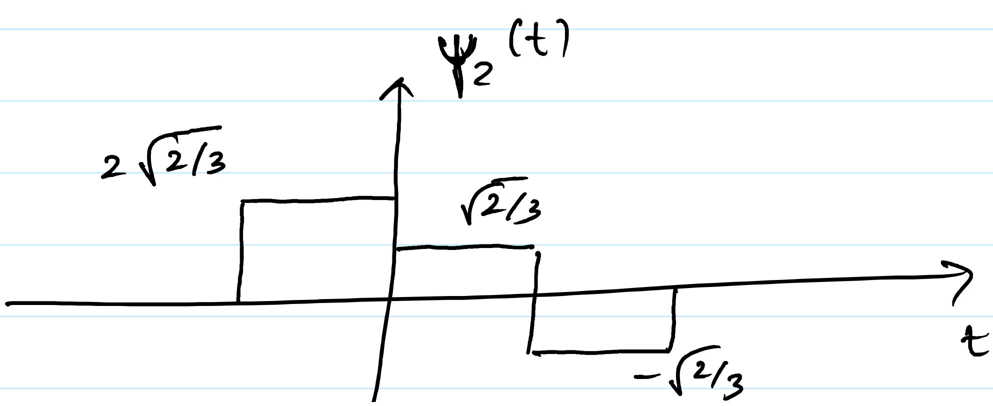
$$\langle s_2(t), \psi_1(t) \rangle = 1/\sqrt{2}$$

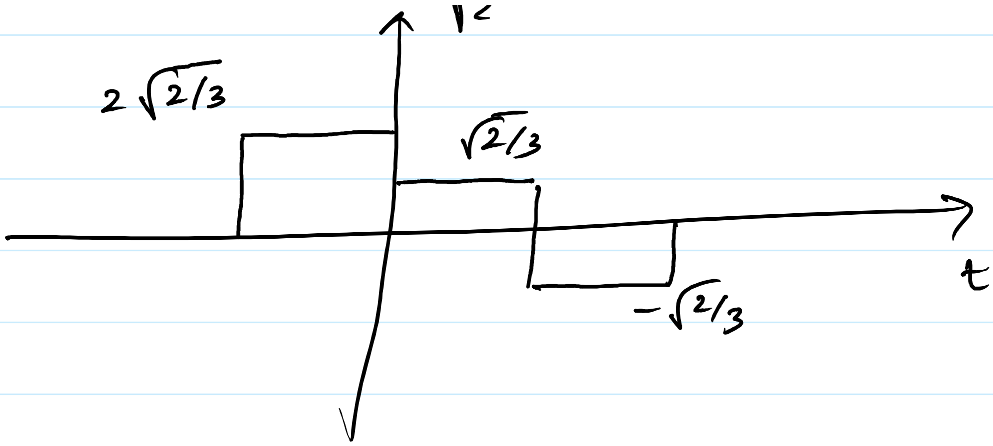
$\phi_2 =$  $-$ 

$=$ 

$$\psi_2(t) = \frac{1}{\|\phi_2\|} \phi_2(t)$$

$$= 2 \cdot \sqrt{2/3} \, \phi_2(t)$$





Step 3

$$\phi_3(t) = s_3(t) - \langle s_3(t), \psi_1(t)\rangle \psi_1(t) - \langle s_3(t), \psi_2(t)\rangle \psi_2(t)$$

Note $\langle s_3(t), \psi_1(t)\rangle = 0$

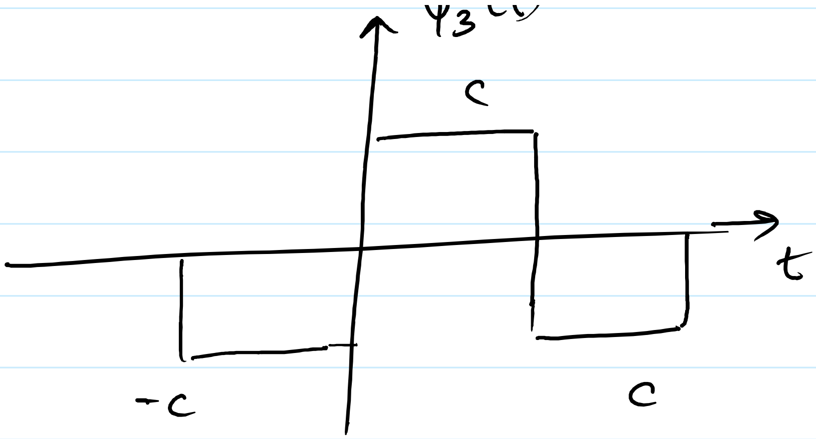
$\langle s_3(t), \psi_2(t)\rangle = 0$

$$\phi_3(t) = s_3(t)$$

$$\psi_3(t) = \frac{s_3(t)}{\| s_3(t) \|}$$

$\psi_3(t)$ $\quad c = \frac{1}{\sqrt{3}}$

c



$\sqrt{3}$

Step 4

$$\phi_4 = s_4(t) - \langle s_4(t), \psi_1(t)\rangle \psi_1(t) - \langle s_4(t), \psi_2(t)\rangle \psi_2(t) - \langle s_4(t), \psi_3(t)\rangle \psi_3(t)$$

Verify that

$$\phi_4(t) = 0$$

No need to update basis set.

x ———— x

Exercise :

Find vector representations of

$s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$

using new basis set.

Verify that

$$\| \underline{s_i} \| = \int_{-\infty}^{\infty} |s_i(t)|^2 dt$$

$$\langle s_i, \underline{s_k} \rangle = \underline{s_k}^* \underline{s_i}$$

$$= \int_{-\infty}^{\infty} s_i(t)\, s_k^*(t)\, dt$$

Signal Space Representations for digital modulated waveforms

Linear Modulation (PSK, PAM, QAM)

$b_1, b_2, \ldots, b_M$ be the

Constellation points

$$s_i(t) = b_i \, p(t), \quad 1 \leq i \leq M$$

Without loss of generality

let $\int_{-\infty}^{\infty} |p(t)|^2 \, dt = 1$

(unit energy pulse)

Signal Set $\mathcal{S} = \{ b_1 \, p(t), b_2 \, p(t), \ldots, b_M \, p(t) \}$

O.N. Basis $\mathcal{B} = \{ p(t) \}$

$$\psi_1(t) = p(t)$$

$\therefore$ single dimensional space

$\underline{s}_1 = [b_1] \quad \ldots \quad \underline{s}_M = [b_M]$

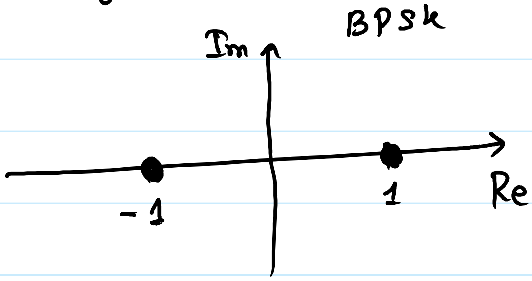
$\underline{s}_2 = [b_2]$

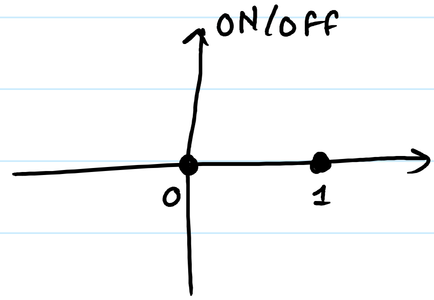
Note b_i can be real (PAM, BPSK) ON-OFF

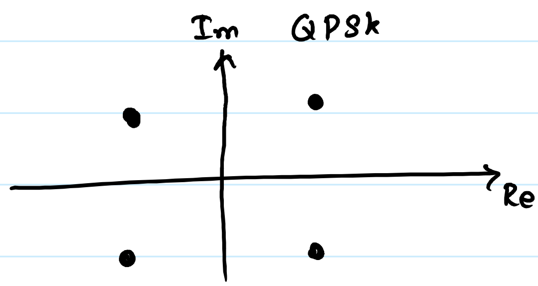
or

complex (QPSK, QAM)

Signal space representation







Signal space representation coincides with constellation (linear modulation)

Nonlinear Modulation (FSK)

M-ary orthogonal FSK

$$s_k(t) = \cos(2\pi f_k t) \; p(t)$$

$f_1, f_2, \ldots, f_M$ are chosen

such that $\{s_k(t)\}$ are orthogonal

Also, without loss of generality,

assume $\|s_k(t)\| = 1$

O.N. Basis for signal space

$$\psi_1(t) = s_1(t)$$

$$\psi_2(t) = s_2(t)$$

$$\vdots$$

$$\psi_M(t) = s_M(t)$$

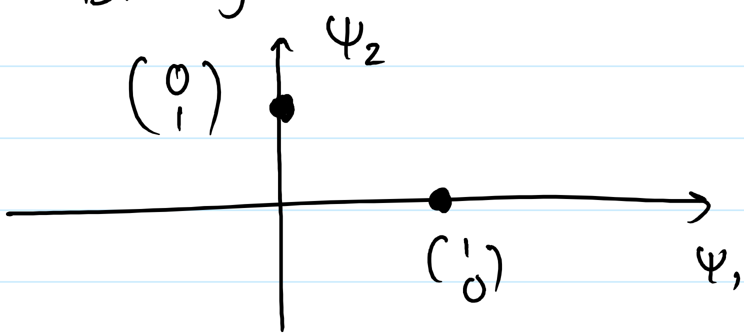
$M \rightarrow$ dimensional space

$$\underline{s}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \underline{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\ldots \qquad \underline{s}_M = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Binary FSK



Vector space representations of signals

→ Easy geometric visualizations

→ Unified framework for studying PSK, QAM, FSK together