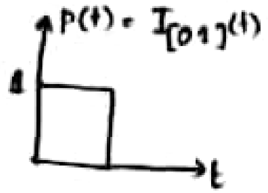
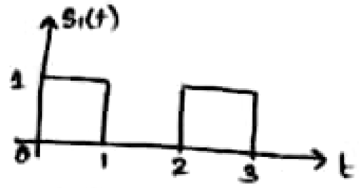
EE5140 – DMC

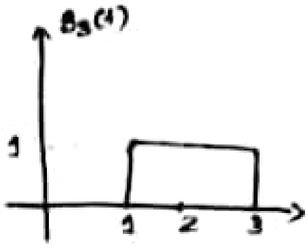
Quiz - 2 (Previous Year solution)

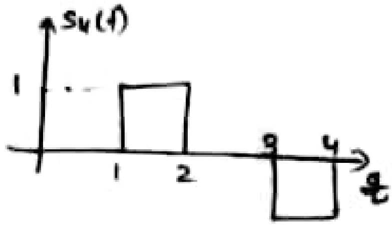
1)



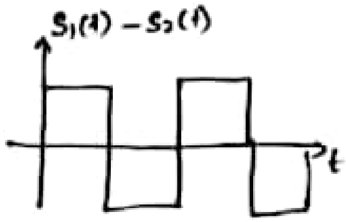





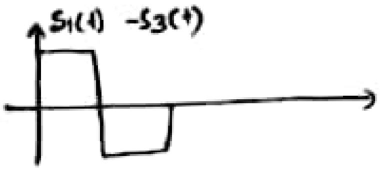


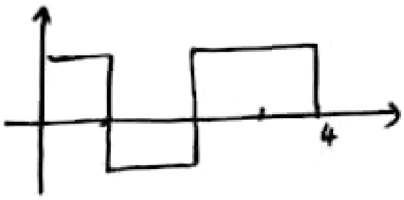

a)



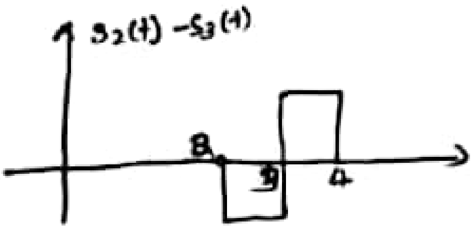

$\|s_1(t) - s_2(t)\|^2 = 4 \times \text{area} = 4\times1\times1 = 4$



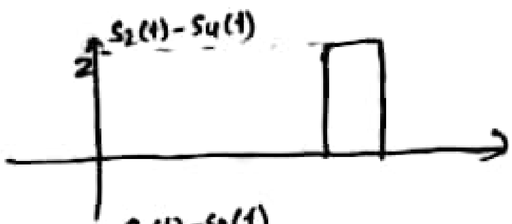

$\|s_1(t) - s_3(t)\|^2 = 2$



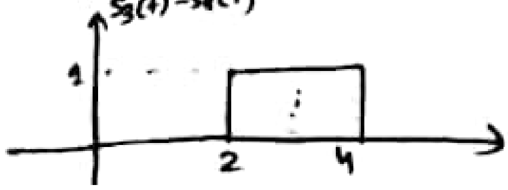

$\|s_1(t) - s_4(t)\|^2 = 4$




$\|s_2(t) - s_3(t)\|^2 = 2$




$\|s_2(t) - s_4(t)\|^2 = 4.$




$\|s_3(t) - s_4(t)\|^2 = 2.$

$$d_{min}^2 = \min_{\substack{p,q \\ p \neq q}} \|s_p(t) - s_q(t)\|^2$$

$$= 2 \qquad \text{(from previow calculations)}$$

$\therefore d_{min} = \sqrt{2}$

b)

$\pi_1 = 0, \ \pi_2 = \pi_3 = \pi_4 = 1/3 \ ; \ \sigma^2 = 10$

$y(t) = 3p(t) + 3p(t-1) - 2p(t-2) + p(t-3)$

Consider the orthonormal basis for the $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$ , $\{\psi_i(t)\}$ such that $\psi_i(t) = p(t-i)$ for $i = 0$ to $3$

Representing in terms of $\{\psi_i(t)\}$

$$\underline{s}_1 = [s_1(t)]_{\psi_i(t)} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \ \underline{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} , \ \underline{s}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} , \ \underline{s}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$y(t) = \underline{y} = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\delta_{MPE}(\underline{y}) = \delta_{MAP}(\underline{y})$$

$$= \underset{1 \le i \le 4}{\operatorname{argmax}} \ \langle \underline{y}, \underline{s}_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i$$

$\left( \|s_i\|^2 = 2 \text{ for all signals} \right)$

$$= \underset{1 \le i \le 4}{\operatorname{argmax}} \ \langle y, \underline{s}_i \rangle + \sigma^2 \log \pi_i$$

For $s_1$ : $\sigma^2 \log \pi_i = \sigma^2 \log 0 = \alpha$ — 1

For $s_2$ : $\langle y, s_2 \rangle + \sigma^2 \log 1/3 = (3+1) + 10 \log 1/3 = 4 + 10 \log 1/3$ —

For $s_3$ : $\langle y, s_3 \rangle + \sigma^2 \log 1/3 = (3-2) + 10 \log 1/3 = 1 + 10 \log 1/3$ — 3

For $s_4$ : $\langle y, s_4 \rangle + \sigma^2 \log 1/3 = (3-1) + 10 \log 1/3 = 2 + 10 \log 1/3$ — 4

Comparing 1 2 3 & 4, maximum is for $s_2$.

$$\Rightarrow \delta_{MPE}(\underline{y}) = \underline{s}_2 = s_2(t)$$

(2) $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2b \\ -4b \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$w_1 \sim N(0, \sigma^2)$

$w_2 \sim N(0, 4\sigma^2)$ , $w_1, w_2$ are iid

Under $H_1$ : $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$(b=1)$

$\underline{m}_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. $C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{bmatrix}$

$\therefore f_{y|H_1}(y) \sim N\left( \begin{bmatrix} 2 \\ -4 \end{bmatrix}, C \right)$

$f_{y|H_1}(y) \sim N(\underline{m}_1, C)$

Under $H_0$ : $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$(b=0)$

$\underline{m}_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{bmatrix}$

$\therefore f_{y|H_0}(y) \sim N\left( \begin{bmatrix} -2 \\ 4 \end{bmatrix}, C \right)$

$f_{y|H_0}(y) \sim N(\underline{m}_0, C)$

We observe that $\underline{m}_0 = -\underline{m}_1$

$\therefore f_{y|H_0}(y) \sim N(-m_1, C)$

Also, $|C| = 4\sigma^4$

$$C^{-1} = \frac{1}{|C|}\begin{bmatrix} 4\sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}^T = \frac{1}{|C|}\begin{bmatrix} 4\sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{4\sigma^2} \end{bmatrix} = \frac{1}{\sigma^2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

ML Rule:

$$\cancel{f(y|H_1)} \quad f_{y|H_1}(y) \underset{H_0}{\overset{H_1}{\gtrless}} f_{y|H_0}(y)$$

$$\frac{1}{2\pi|C|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y}-\underline{m_1})^T C^{-1}(\underline{y}-\underline{m_1})\right\} \underset{H_0}{\overset{H_1}{\gtrless}}$$

$$\frac{1}{2\pi|C|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y}-\underline{m_0})^T C^{-1}(\underline{y}-\underline{m_0})\right\}$$

$$\Rightarrow \quad -\frac{1}{2}(\underline{y}-\underline{m_1})^T C^{-1}(\underline{y}-\underline{m_1}) \underset{H_0}{\overset{H_1}{\gtrless}} -\frac{1}{2}(\underline{y}+\underline{m_1})^T C^{-1}(\underline{y}+\underline{m_1})$$

$$\Rightarrow \quad (\underline{y}-\underline{m_1})^T C^{-1}(\underline{y}-\underline{m_1}) \underset{H_1}{\overset{H_0}{\gtrless}} (\underline{y}+\underline{m_1})^T C^{-1}(\underline{y}+\underline{m_1})$$

OR

$$\cancel{\underline{y}^T C^{-1}\underline{y}} - \underline{y}^T C^{-1}\underline{m_1} - \underline{m_1}^T C^{-1}\underline{y} + \cancel{\underline{m_1}^T C^{-1}\underline{m_1}} \underset{H_1}{\overset{H_0}{\gtrless}}$$

$$\cancel{\underline{y}^T C^{-1}\underline{y}} + \underline{y}^T C^{-1}\underline{m_1} + \underline{m_1}^T C^{-1}\underline{y} + \cancel{\underline{m_1}^T C^{-1}\underline{m_1}}$$

$$\cancel{-2\underline{y}^T C^{-1}\underline{m_1} \gtrless 2\underline{y}^T C^{-1} m}$$

$$-2\underline{y}^T C^{-1}\underline{m_1} - 2\underline{m_1}^T C^{-1}\underline{y} \underset{H_1}{\overset{H_0}{\gtrless}} 0$$

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$$\underline{y}^T C^{-1} \underline{m}_1 + \underline{m}_1^T C^{-1} \underline{y} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$[y_1 \; y_2] \frac{1}{6^2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} + [2 \; -4] \frac{1}{6^2}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$[y_1 \; y_2] \begin{bmatrix} 2 \\ -1 \end{bmatrix} + [2 \; -1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

or $2y_1 - y_2 + 2y_1 - y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$

$$4y_1 - 2y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

or $y_1 - \frac{1}{2}y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$

OR, $\hat{b} = \begin{cases} 1 & \text{if } y_1 - \frac{1}{2}y_2 > 0 \\ -1 & \text{if } y_1 - \frac{1}{2}y_2 < 0 \end{cases}$

$\therefore \alpha = -\frac{1}{2}$