Performance Analysis

10 October 2016 09:25

We compute (average) probability of error for different modulation schemes under optimal coherent demodulation

We assume (for convenience & also valid in practice) all prior probabilities are same

$$\pi_1 = \pi_2 = \cdots = \pi_M = 1/M$$

So $\delta_{ML} = \delta_{MAP} = \delta_{MPE}$

Binary Signalling (real case)

Under H_0 : $y(t) = s_0(t) + \underbrace{w(t)}_{\text{WGN}}$

H_1 : $y(t) = s_1(t) + w(t)$

MPE Rule is ML rule ($\pi_0 = \pi_1$)

$$\delta_{ML} = \arg\max_{i=0,1} \langle y(t), s_i(t)\rangle - \frac{\|s_i(t)\|^2}{2}$$

Equivalently

$$\langle y(t), s_1(t)\rangle - \frac{\|s_1(t)\|^2}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \langle y(t), s_0(t)\rangle - \frac{\|s_0(t)\|^2}{2}$$

$$\underbrace{\langle y(t), s_1(t)\rangle - \langle y(t), s_0(t)\rangle}_{Z = \langle y(t),\ s_1(t) - s_0(t)\rangle} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\|s_1(t)\|^2}{2} - \frac{\|s_0(t)\|^2}{2}$$

$\downarrow$ z is Gaussian random variable

z is called decision statistic

Conditional Prob. of error

$P_{e|1} \Rightarrow$ Prob. of Error given H_1 is true

$$P_{e|1} = P_r\left\{ z < \frac{\|s_1(t)\|^2 - \|s_0(t)\|^2}{2} \,\middle|\, H_1 \right\}$$

$\downarrow$

To compute this we need mean & variance of z under H_1

Under H_1: $\quad y(t) = s_1(t) + w(t)$

$$z = \langle y(t),\ s_1(t) - s_0(t) \rangle$$

$$= \langle s_1(t) + w(t),\ s_1(t) - s_0(t) \rangle$$

$$z = \langle s_1(t),\ s_1(t) - s_0(t) \rangle + \underbrace{\langle w(t),\ s_1(t) - s_0(t) \rangle}_{\substack{0 \text{ mean} \\ \text{Var. } \sigma^2 \|s_1(t) - s_0(t)\|^2}}$$

$$z \sim N\Big(\underbrace{\langle s_1(t),\ s_1(t) - s_0(t) \rangle}_{\text{Mean of } z},\ \sigma^2 \|s_1(t) - s_0(t)\|^2 \Big)$$

$$E(z) = \langle s_1(t),\ s_1(t) - s_0(t) \rangle$$

$$= \|s_1(t)\|^2 - \langle s_1(t),\ s_0(t) \rangle$$

$$\mathrm{Var}(z) = \sigma^2 \|s_1(t) - s_0(t)\|^2$$

$$P_{e|1} = P_r\left\{ z < \frac{\|s_1(t)\|^2 - \|s_0(t)\|^2}{2} \right\}$$

Recall $Q(x) = \underbrace{\int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt}_{P_r\{N(0,1) > x\}}$

Define $$\tilde{z} = \frac{z - E(z)}{\sqrt{Var(z)}}$$

$\tilde{z} \sim N(0,1)$

$$\tilde{z} = \frac{z - \|s_1(t)\|^2 + \langle s_1(t), s_0(t)\rangle}{\sigma \|s_1(t) - s_0(t)\|}$$

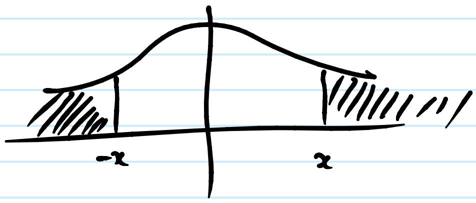
$$P_{e|1} = P_r\left\{ z < \frac{\|s_1(t)\|^2 - \|s_0(t)\|^2}{2} \right\}$$

$$= P_r\left\{ \tilde{z} < \frac{\left(\frac{\|s_1(t)\|^2}{2} - \frac{\|s_0(t)\|^2}{2} - \|s_1(t)\|^2 + \langle s_0, s_1\rangle \right)}{\sigma \|s_1(t) - s_0(t)\|} \right\}$$

$\|s_1(t) - s_0(t)\|^2$

$= \|s_1(t)\|^2 + \|s_0(t)\|^2 - 2\langle s_1(t), s_0(t)\rangle$

$$= P_r\left\{ \tilde{z} < -\frac{1}{2\sigma} \|s_1(t) - s_0(t)\| \right\}$$



$$= P_r\left\{ \tilde{z} > \frac{1}{2\sigma} \|s_1(t) - s_0(t)\| \right\}$$

$$P_{e|1} = Q\left(\frac{\|s_1(t) - s_0(t)\|}{2\sigma} \right)$$

Similarly

$\|s_1(t) - s_0(t)\| \rightarrow$ Euclidean distance between $s_1(t)$ & $s_0(t)$

$$P_{e|0} = Q\left(\frac{\|s_1(t) - s_0(t)\|}{2\sigma} \right)$$

Avg. Prob of Error $P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0}$

$$= Q\left(\frac{\|s_1(t) - s_0(t)\|}{2\sigma}\right)$$

$$d = \|s_1(t) - s_0(t)\|$$

$$P_e = Q\left(\frac{d}{2\sigma}\right)$$

Some Useful definitions.

E_b = Energy per bit = Average Energy Spent in transmitting a bit

Energy Spent if bit is 0 $= \|s_0(t)\|^2$

Energy Spent if bit is 1 $= \|s_1(t)\|^2$

$$E_b = \text{Avg Energy} = \frac{\|s_1(t)\|^2 + \|s_0(t)\|^2}{2}$$

Noise power spectral density (recall)

(definition) $\frac{N_0}{2} = \sigma^2$

Power Efficiency $\eta_P = \frac{d^2}{E_b} = \frac{\|s_1(t) - s_0(t)\|^2}{E_b}$

η_P is invariant to <u>Scaling</u>

Suppose $\tilde{s}_1(t) = \alpha\, s_1(t)$

$\tilde{s}_0(t) = \alpha\, s_0(t)$

$$\tilde{d} = \|\tilde{s}_1(t) - \tilde{s}_0(t)\|$$

$$= \alpha\, \|s_1(t) - s_0(t)\|$$

$$\tilde{E}_b = \alpha^2 E_b$$

$$\tilde{\eta}_P = \frac{\tilde{d}^2}{\tilde{E}_b} = \frac{d^2}{E_b} = \eta_P$$

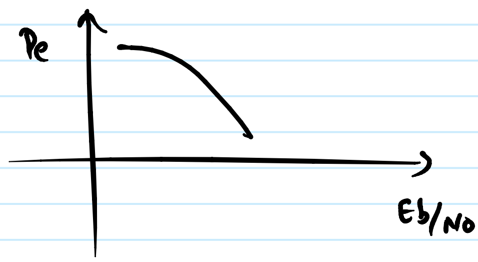
Now Prob. of Error P_e can be written as

$$P_e = Q\left(\frac{d}{2\sigma}\right)$$

$$= Q\left(\sqrt{\frac{d^2}{4\sigma^2}}\right)$$

$$= Q\left(\sqrt{\frac{d^2}{E_b} \cdot \frac{E_b}{2N_0}}\right)$$

$$\boxed{P_e = Q\sqrt{\left(\frac{\eta_P}{2}\right)\frac{E_b}{N_0}}}$$



Notes :

- P_e depends only on ratio $\frac{E_b}{N_0}$
- Actual signals $s_1(t), s_0(t)$ affect the power efficiency η_P
- Higher $\eta_P \Rightarrow$ Lower P_e

x ———— x

P_e for common binary signalling schemes (η_P)

- $\eta_P = \frac{d^2}{E_b}$

  We can compute distances using appropriate signal space representation

- We can choose "convenient" scaling

Example:

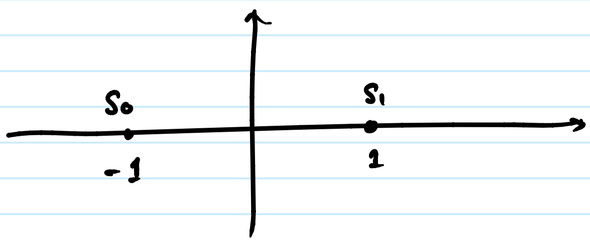
① BPSK Signalling

$$s_1(t) = p(t)$$
$$s_0(t) = -p(t)$$

Orthonormal Basis $\quad \varphi_1(t) = \dfrac{p(t)}{\|p(t)\|} = \alpha\, p(t)$

$s_1(t) = \frac{1}{\alpha}\varphi_1(t)$

$s_0(t) = -\frac{1}{\alpha}\varphi_1(t)$



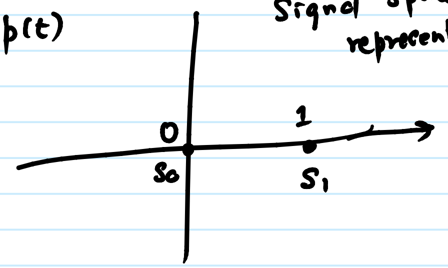
$d = 2 \qquad E_b = 1 \qquad \eta_p = 4$

$$P_e(\text{BPSK}) = Q\sqrt{\frac{2E_b}{N_0}}$$

② ON-OFF Signalling

$$s_0(t) = 0$$
$$s_1(t) = p(t)$$

Signal Space representation



$d = 1$

$E_b = \frac{0}{2} + \frac{1}{2} = \frac{1}{2} \qquad \eta_p = 2$

$$P_e(\text{ON-OFF}) = Q\sqrt{\frac{E_b}{N_0}}$$

③ Binary Orthogonal FSK (Equal Energy)

$$s_1(t) = \cos 2\pi f_1 t \qquad 0 \le t \le T$$
$$s_0(t) = \cos 2\pi f_0 t$$

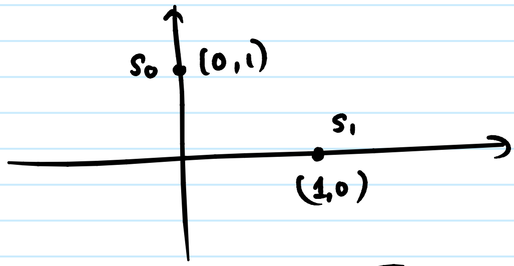
f_1, f_0

such that $\|s_1(t)\| = \|s_0(t)\|$

$\langle s_1(t), s_0(t)\rangle = 0$

$\varphi_1(t) = \alpha\, s_1(t)$

$\varphi_2(t) = \alpha\, s_0(t)$



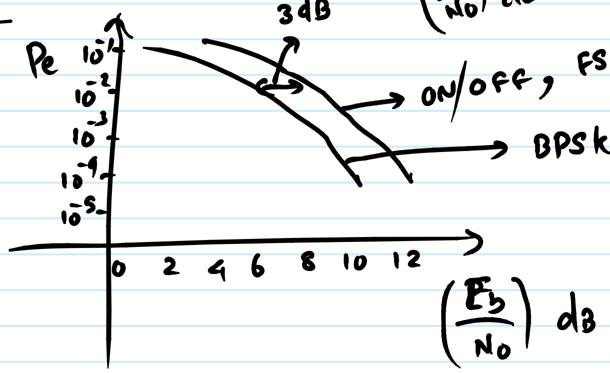
Compute $\eta_P = 2$ $\qquad d = \sqrt{2}$

$E_b = 1$

$$P_e = Q\sqrt{\frac{E_b}{N_0}}$$

Remarks.

$$\left(\frac{E_b}{N_0}\right)_{dB} = 10\log_{10}\frac{E_b}{N_0}.$$



Remark

* ON/OFF (Unequal energy orthogonal) & FSK (equal energy orthogonal) has same error performance

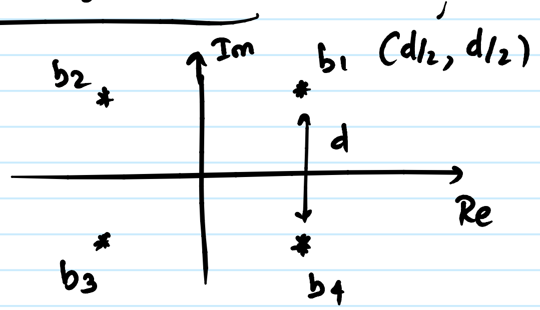
* BPSK Signalling (Antipodal) has (3 dB) better performance than ON/OFF, FSK

$10\log 2 \approx 3$ dB

Error Performance of QPSK

Error Performance of QPSK

$M = 4$



$b_i \rightarrow i^{th}$ Constellation point

$$s_i(t) = Re\{ b_i \, p(t) \, e^{j2\pi f_c t} \}$$

$p(t) \rightarrow$ pulse shape

$$= Re\{b_i\} \, p(t) \cos 2\pi f_c t - Im(b_i) \, p(t) \sin 2\pi f_c t$$

Orthonormal basis

$$\psi_1(t) = \alpha \, p(t) \cos 2\pi f_c t$$

$\alpha \rightarrow$ to normalize for unit norm

$$\psi_2(t) = \alpha \, p(t) \sin 2\pi f_c t$$

Under H_i : $y(t) = s_i(t) + \underbrace{w(t)}_{\text{WGN}}$ Variance σ^2

$$y_I = \langle y(t), \psi_1(t) \rangle$$

$$y_Q = \langle y(t), \psi_2(t) \rangle$$

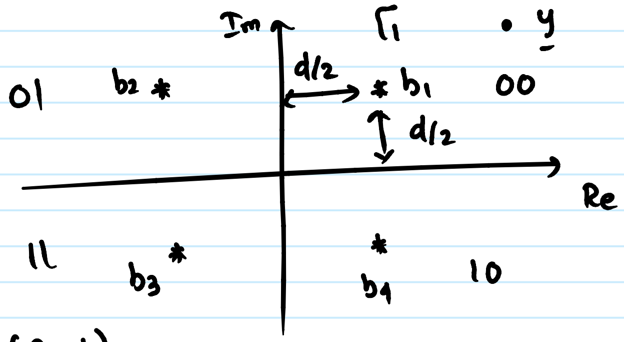
$$y_I = \underbrace{\langle s_i(t), \psi_1(t) \rangle}_{} + \langle w(t), \psi_1(t) \rangle$$

$$= Re\{b_i\} + \underbrace{w_I}_{N(0,\sigma^2)}$$

$$y_Q = Im\{b_i\} + \underbrace{w_Q}_{N(0,\sigma^2)}$$

w_I & w_Q are independent

$$\underline{y} = \begin{bmatrix} y_I \\ y_Q \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{b_i\} \\ \mathrm{Im}\{b_i\} \end{bmatrix} + \begin{bmatrix} W_I \\ W_Q \end{bmatrix}$$



$P_{e|i}$ = (Cond) Prob. that error happens given we send b_i

$$P_e = \sum_{i=1}^{4} \pi_i P_{e|i}$$ ($\pi_i \rightarrow$ prior prob. for sending b_i)

If $\pi_1 = \pi_2 = \pi_3 = \pi_4$ then

$$P_e = \frac{\sum P_{e|i}}{4}$$

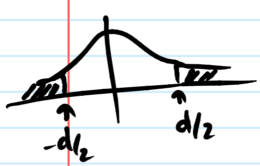
$P_{e|1}$ = Cond. Prob. of Error given we send b_1

$$= \Pr\{\underline{y} \notin \Gamma_1 \text{ (first quadrant)} \mid b_1 \text{ is Sent}\}$$

$$= \Pr\{W_I < -d/2 \text{ (OR) } W_Q < -d/2\}$$

$$= \Pr\{W_I < -d/2\} + \Pr\{W_Q < -d/2\} - \Pr\{W_I < -d/2 \ \& \ W_Q < -d/2\}$$

$$\boxed{\begin{aligned} &P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}}$$



$$= \Pr\{W_I > d/2\} + \Pr\{W_Q > d/2\}$$

$$- \Pr\{W_I > d/2 \ \& \ W_Q > d/2\}$$

$= \Pr\{w_I > d/2\} + \Pr\{w_Q > d/2\}$

$$- \Pr\{w_I > d/2 \ \& \ w_Q > d/2\}$$

$$\Pr\{w_I > d/2\} \cdot \Pr\{w_Q > d/2\}$$

$$= Q\left(\frac{d}{2\sigma}\right) + Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{d}{2\sigma}\right) \cdot Q\left(\frac{d}{2\sigma}\right)$$

$$= 2\,Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right)$$

Because of Symmetry in constellation,

Conditional error probabilities are Same

$$P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$$

Avg. Prob. of error $P_e = 2\,Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right)$

Let us write P_e in ~~terms of~~ E_b/N_0.

$$\sigma^2 = N_0/2$$

To find E_b:

$$E_b = \frac{\text{Averge Symbol Energy } (E_s)}{2}$$

QPSK $\Rightarrow$ 2 bits per symbol

Average Symbol Energy E_s

$$= \frac{|b_1|^2 + |b_2|^2 + |b_3|^2 + |b_4|^2}{4}$$

$$|b_i|^2 = (d/2)^2 + (d/2)^2 = d^2/2$$

$$E_s = d^2/2.$$

$\frac{N_0}{2} = \sigma^2$

$$E_b = E_s/2 = d^2/4$$

$$\frac{d}{2\sigma} = \sqrt{\frac{2E_b}{N_0}}$$

$$P_e(QPSK) = 2Q\sqrt{\frac{2E_b}{N_0}} - Q^2\sqrt{\frac{2E_b}{N_0}}$$

Prob. of Symbol error

x ———— x

Performance of M-ary Signalling

General value of M

Under H_i : $y(t) = s_i(t) + w(t)$

$i = 1$ to M

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\arg\max} \; \langle y(t), s_i(t)\rangle - \frac{\|s_i(t)\|^2}{2}$$

$1 \le i \le M$; $Z_i = \langle y(t), s_i(t)\rangle - \frac{\|s_i(t)\|^2}{2}$

Joint Statistics of $\{Z_1, Z_2, \ldots, Z_M\}$

Under H_k {transmitted Signal is $S_k(t)$}

$$y(t) = S_k(t) + w(t)$$

$$Z_i = \langle y(t), s_i(t)\rangle - \frac{\|s_i(t)\|^2}{2}$$

Verify Z_i is Gaussian

$$E(Z_i) = \langle s_k(t), s_i(t)\rangle - \frac{\|s_i(t)\|^2}{2}$$

$$Var(Z_i) = \sigma^2 \|s_i(t)\|^2$$

$$Cov(Z_l, Z_m) = E\left(\left(Z_l - E(Z_l)\right)\left(Z_m - E(Z_m)\right)\right)$$

$$\mathrm{Cov}(z_\ell, z_m) = E\left(\left(z_\ell - E(z_\ell)\right)\left(z_m - E(z_m)\right)\right)$$
$$= \sigma^2 \langle s_\ell(t), s_m(t) \rangle$$

Conditional Prob. of Error

$P_{e|k}$ = Prob. of Error given $s_k(t)$ is the transmitted signal

$= \Pr\{ z_k <$ atleast one of $z_i \;\; i \neq k \}$

$$= \Pr\left\{ \bigcup_{i \neq k} z_k < z_i \,\middle|\, H_k \right\}$$

Multi-dim. Gaussian pdf integration

Union bound

$P(A \cup B) \leq P(A) + P(B)$

$$P_{e|k} \leq \sum_{i \neq k} \Pr\{ z_k < z_i \mid H_k \}$$

$P\{ z_k < z_i \mid H_k \} \rightarrow$ error probab. of binary signalling with signals $s_k(t)$ & $s_i(t)$

Pairwise error prob $P(s_k \rightarrow s_i)$ (Notation)

We have already computed P_e for binary signalling

$$P\{ z_k < z_i \mid H_k \} = Q\left(\frac{\| s_k(t) - s_i(t) \|}{2\sigma} \right)$$

Let $d_{ij} = \| s_i(t) - s_j(t) \|$

$$P_{e|k} \le \sum_{i \ne k} Q\left(\frac{d_{ik}}{2\sigma}\right)$$

Avg. Prob. of Error

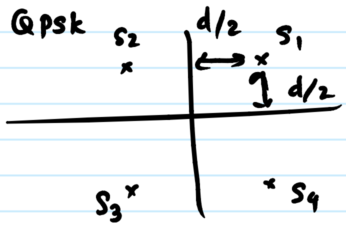
$$P_e = \sum_{k=1}^{M} \pi_k P_{e|k}$$

$$P_e \le \sum_{k=1}^{M} \pi_k \sum_{i \ne k} Q\left(\frac{d_{ik}}{2\sigma}\right)$$

This upper bound on P_e is called union bound on P_e

x ———— x

Example QPSK



$$P_e = P_{e|s_1} \le Q\left(\frac{d_{14}}{2\sigma}\right) + Q\left(\frac{d_{13}}{2\sigma}\right) + Q\left(\frac{d_{12}}{2\sigma}\right)$$

$$d_{14} = d = d_{12} \; ; \quad d_{13} = \sqrt{2}\, d$$

dominant

$$P_e \le 2\, Q\left(\frac{d}{2\sigma}\right) + Q\left(\frac{\sqrt{2}\, d}{2\sigma}\right)$$

you can write in terms of E_b/N_0

x ———— x

Actual P_e for QPSK

dominant

$$P_e(\text{QPSK}) = 2\, Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right)$$

For large values of x, $Q(x) \approx e^{-x^2/2}$

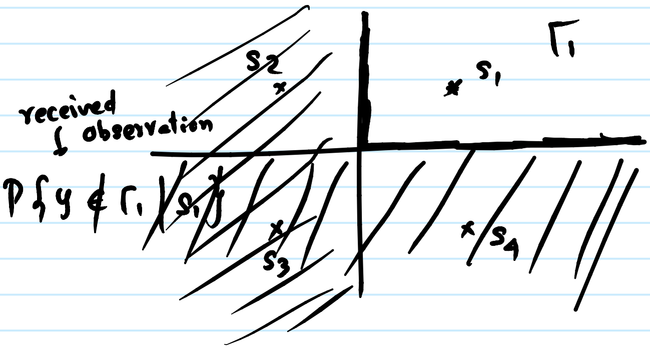
(decays exponentially)

Union bound gets the multiplicity of the argument of dominant term correctly.

x ———————— x

Intelligent Union bound

→ Improves union bound by pruning terms in the ~~bound~~



$Pe|_2 = P\{y \notin \Gamma_1 | s_1\}$

Pairwise Error Prob (PEP)

$P\{s_1 \to s_4\} = P\{y$ is closer to s_4 than $s_1 | s_1\}$

→ y below real axis

$P\{s_1 \to s_2\} = P\{y$ is closer to s_2 than $s_1 | s_1\}$

→ y left of imag axis

We have

$$Pe|_{s_1} \leq P\{s_1 \to s_2\} + P\{s_1 \to s_4\}$$

$P(s_1 \to s_3) \Rightarrow$ is not needed in the upper bound.

s_2 & s_4 are nearest neighbors for s_1

ie) they define the region Γ_1

(s_3 does not affect the decision region Γ_1)

x ———————— x

For a general Constellation

$P_{e|i} \rightarrow$ Prob of error given S_i is sent

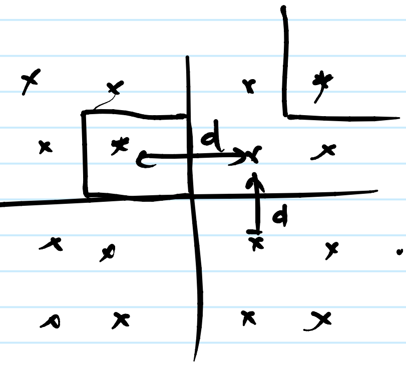
$N(i) \rightarrow$ set of neighbors for S_i

$\Downarrow$

ones which define Γ_i

$$P_{e|i} \leq \sum_{j \in N(i)} P(S_i \rightarrow S_j)$$

$$\sum_{j \in N(i)} Q\left(\frac{d_{ij}}{2\sigma}\right)$$



intelligent union bound

16 QAM

Find intelligent union bound for 16QAM

Nearest Neighbour Approximation

$$d_{min} = \min_{i \neq j} \| S_i - S_j \|$$

$N_{min}(i) \rightarrow$ set of nearest neighbours at distance d_{min} from S_i

We approximate $P_{e|i}$ as

$$P_{e|i} \approx N_{min}(i)\, Q\left(\frac{d_{min}}{2\sigma}\right)$$

$$P_e = \frac{1}{M} \sum_{i=1}^{M} P_{e|i} \qquad (\pi_1 = \pi_2 = \dots = \pi_M)$$

$$P_e \approx \overline{N_{min}}\, Q\left(\frac{d_{min}}{2\sigma}\right)$$

$$\overline{N_{min}} = \frac{1}{M} \sum^{M} N_{min}(i)$$

$$\overline{N_{min}} = \frac{1}{M} \sum_{i=1}^{M} N_{min}(i)$$

Define Power Efficiency (η_p)

for general Constellation

$$\eta_p = \frac{d_{min}^2}{E_b} \qquad \sigma^2 = \frac{N_0}{2}$$

Nearest Neighbour Approx

$$P_e \approx \overline{N_{min}}\, Q\sqrt{\frac{\eta_p}{2}\frac{E_b}{N_0}}$$

For QPSK $\eta_p = 4$

For 16-QAM $\eta_p = 1.6$

$$P_e(\text{16 QAM}) \approx 3\, Q\sqrt{\frac{4}{5}\frac{E_b}{N_0}}$$

Nearest Neighbour Approx.

For M-QAM (and M psk)

η_p decreases with increase in M.

Bandwidth efficiency $\eta_B = \frac{\text{Bit rate}}{\text{Bandwidth}}$

Symbol rate $= 1/T$

RC pulse with excess BW parameter a

RC pulse with excess BW parameter a

$$BW = (1+a)\frac{1}{T}$$

bit rate (M-ary Signalling)

$$= \frac{\log M}{T}$$

$$\eta_B = \frac{\log M/T}{(1+a)/T} = \frac{\log M}{(1+a)}$$

For M QAM (MPSK) as

M increases η_B increases

η_P decreases

x ———————— x

Performance of M-ary Orthogonal Signalling (M-FSK)

M - orthogonal Signals

Signal - Space representation

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad s_M = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$d_{min} = \sqrt{2} = \| s_i - s_j \|$$

$$P(s_i \rightarrow s_j) = Q\left(\frac{d_{ij}}{2\sigma}\right) = Q\left(\frac{d_{min}}{2\sigma}\right)$$

$$\eta_P = \frac{d_{min}^2}{E_b}$$

$$E_b = \frac{E_s}{\log M}$$ → Avg. ~~symbol~~ power

$E_s = 1$

Since $\|s_i\| = 1$ $\forall i$

$$= \frac{1}{\log M}$$

$\forall i$

$$= \frac{1}{\log M}$$

$$P(s_i \to s_j) = Q\sqrt{\frac{\eta_p}{2} \frac{E_b}{N_o}}$$

$$= Q\sqrt{\frac{E_b \log M}{N_o}}$$

$$P_{e|i} \underbrace{\leq}_{\text{union bound}} \sum_{i \neq j} P(s_i \to s_j)$$

$$P_{e|i} \leq (M-1)\, Q\sqrt{\log M \frac{E_b}{N_o}}$$

$$\text{Avg. } P_e \underbrace{\leq}_{\text{(upper) union bound}} (M-1)\, Q\sqrt{\frac{E_b}{N_o} \log M}$$

$$(M-1)\, Q\sqrt{\frac{E_b}{N_o} (\log M)}$$

As $M \to \infty$ $\quad (M-1) \to \infty$

$$Q\sqrt{\frac{E_b}{N_o} \log M} \longrightarrow 0$$

Use L'Hospitals rule

$$Q(x) = \int_x^\infty e^{-t^2/2}\, dt$$

$$\frac{d}{dx} Q(x) = e^{-x^2/2}$$

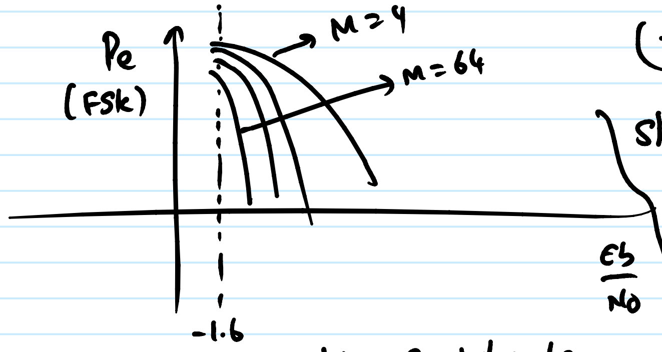
$$(M-1)\, Q\sqrt{\frac{E_b}{N_o} \log_2 M} \longrightarrow 0$$

if $\frac{E_b}{N_o} > 2 \ln 2$

(Upper bound on $P_e \to 0$ if $\frac{E_b}{N_o} > 2 \ln 2$

As $M \to \infty$

- Upper bound on $P_e \to 0$ if $\frac{E_b}{N_0} > 2\ln 2$
- Actual $P_e \to 0$ if $\frac{E_b}{N_0} > \ln 2$ (-1.6 dB) $\Downarrow$ Shannon's limit for reliable communication)



$1/T$ Symbol rate

Orthogonal FSK (min) freq spacing is $\frac{1}{2T}$ (coherent demod)

$$M\text{-FSK} \quad BW = M\left(\frac{1}{2T}\right)$$

$$\text{Bit rate} = \log_2 M / T$$

$$\eta_B = \frac{2\log M}{M}$$

$$\eta_P = 2\log M = \frac{d_{min}^2}{E_b}$$

As $M \uparrow$ $\quad \eta_B \downarrow \quad \eta_P \uparrow$

x ——— x

Section 3.7 (Link Budget Analysis)

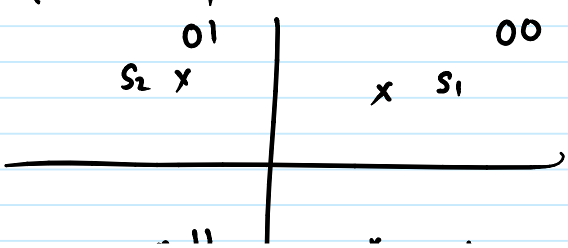
x ——— x

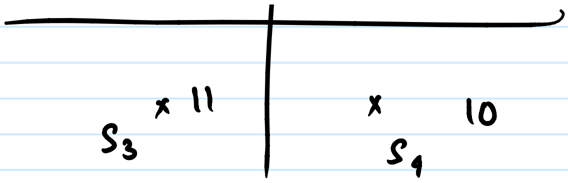
Bit level Demodulation

Consider QPSK example

$M = 4$

2 bits/symbol





$S_i \to i^{th}$ symbol

$a_{i0}\ a_{i1} \to$ bits for i^{th} symbol

So far we looked at optimal symbol demodulation

MAP for symbol demodulation

$$\hat{S_i} = \delta_{MAP}(y) = \arg\max_i P(S_i|y)$$

minimizes probability of symbol error

Say

$$P(S_1|y) = 0.35$$
$$P(S_2|y) = 0.1$$
$$P(S_3|y) = 0.3$$
$$P(S_4|y) = 0.25$$

MAP decoder output is S_1

corresponding decoded bits is 00

We can decode bits directly as follows

<u>MAP for bit demodulation</u>

$$a_{i0}, a_{i1}$$

$a_{i0} \to$ takes 0 or 1

MAP demod for bit a_{i0} →

$$P(a_{i0} = 0|y) \underset{H_1}{\overset{H_0}{\gtrless}} P(a_{i0} = 1|y)$$

$$P(a_{i1} = 0|y) \underset{H_1}{\overset{H_0}{\gtrless}} P(a_{i1} = 1|y)$$

From the mapping,

From the mapping,

$$P(a_{i0}=0 \mid y) = P(S_1 \text{ or } S_2 \mid y)$$

$$= P(S_1|y) + P(S_2|y)$$

$$= 0.35 + 0.1 = 0.45$$

$$P(a_{i0}=1 \mid y) = P(S_3|y) + P(S_4|y) = 0.55$$

MAP Demod output for first bit is 1

Similarly MAP demod output for second bit is 0

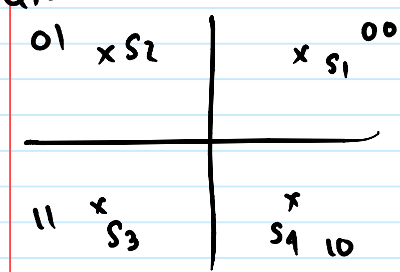
MAP bit demod is different from MAP Symbol demod

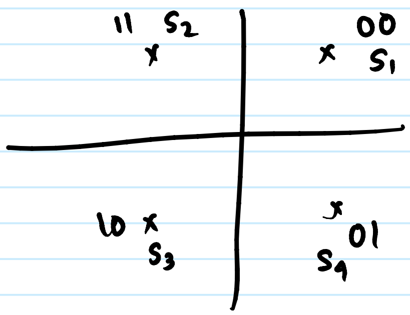
MAP bit demod → minimizes bit error rate (BER)

MAP Symbol demod → minimizes symbol error rate (SER)

BER depends on how bits get mapped to symbols. (with MAP symbol demod)

QPSK





GRAY Coding

Nearest neighbours have only one bit difference

We can compute Prob. of bit error

We can compute Prob. of bit error
for MAP bit demod

(book)

x