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# Non Coherent Demodulation

$x_p(t) \rightarrow$ passband transmit signal

$x(t) \rightarrow$ baseband complex envelope

$f_c \rightarrow$ carrier freq.

$$x_p(t) = \text{Re}\left\{ x(t)\, e^{j2\pi f_c t} \right\}$$

~~A~~ received signal is delay / attenuation of transmit signal with noise

$$y_p(t) = A\, x_p(t-\tau) + \underbrace{w_p(t)}_{\text{noise}}$$

$A \rightarrow$ Attenuation ; delay $\tau$

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$$x_p(t-\tau) = \mathrm{Re}\left\{ x(t-\tau)\, e^{j2\pi f_c (t-\tau)} \right\}$$

$$= \mathrm{Re}\left\{ x(t-\tau)\, e^{-j2\pi f_c \tau}\, e^{j2\pi f_c t} \right\}$$

$$\theta = (-2\pi f_c \tau) \bmod (2\pi)$$

$y_p(t)$ convert to base band

$\Downarrow$

$$y(t) = A\, x(t-\tau)\, e^{j\theta} + \underbrace{w(t)}_{\text{baseband noise}}$$

$\theta \rightarrow$ very sensitive to delay $\tau$

since $f_c$ is quite high

$\theta$ is unknown and $\theta \in [0, 2\pi]$

It is also common to model $\theta$ as random with uniform distribution in $[0, 2\pi]$

Noncoherent Comm. model

(Complex baseband)

$M$ possible symbols $(s_1, s_2, \ldots, s_M)$

Under $H_i$ ; $y(t) = s_i(t)\, e^{j\theta} + w(t)$

$\theta \rightarrow$ unknown  $s_i(t) \rightarrow$ known

We arrive at the same model

if we upconvert using $e^{j(2\pi f_c t + \phi_1)}$

$-j(2\pi f_c t + \phi_2)$

& down convert using $e^{-j(2\pi f_c t + \phi_2)}$

Here $\theta = \phi_1 - \phi_2$

We need $f(y|H_i)$ to do hypothesis testing

- We assume $\theta$ is random with uniform distribution in $[0, 2\pi]$

It is easy to get $f(y|H_i;\theta)$

Now $f(y|H_i) = \int_{-\infty}^{\infty} f(y|H_i;\theta) \cdot f(\theta|H_i)\, d\theta$

$2\pi$

$$= \int_0 f(y|H_i,\theta) \cdot \frac{1}{2\pi} \, d\theta$$

We start with Simpler model (Real Case)

Signal Present $\rightarrow$ $H_\theta$ : $y(t) = S_\theta(t) + w(t)$

Signal absent $\rightarrow$ $H_n$ : $y(t) = w(t)$

$S_\theta(t) \rightarrow$ Signal with some parameter $\theta$

$w(t) \rightarrow$ white Gaussian noise

Suppose $\underline{\underline{\theta \text{ is}}}$ given

Let $P = \langle y(t), S_\theta(t) \rangle$

sufficient statistic for hypothesis testing

$$H_\theta : \quad P \sim N\left( \|s_\theta(t)\|^2 , \sigma^2 \|s_\theta(t)\|^2 \right)$$

$$H_n : \quad P \sim N\left( 0 , \sigma^2 \|s_\theta(t)\|^2 \right)$$

Define likelihood function

$$L(y|\theta) = \frac{f(P|H_\theta)}{f(P|H_n)}$$

$$= e^{\left\{ \frac{1}{\sigma^2} \left\{ P - \frac{\|s_\theta\|^2}{2} \right\} \right\}} .$$

$$P = \langle y(t), s_\theta(t) \rangle$$

for real case

For complex baseband

For complex base band

Suppose ~~<y(t)>~~ $y(t)$ & $S_\theta(t)$ are complex baseband

$$L(y|\theta) = e^{\frac{1}{2\sigma^2}\left\{ \mathrm{Re}\langle y(t), S_\theta(t)\rangle - \frac{\| S_\theta(t) \|^2}{2} \right\}}$$

M-ary Signalling with unknown phase

$$H_i : \quad y(t) = S_i(t)\, e^{j\theta} + w(t)$$

$$L(y|H_i,\theta) = e^{\frac{1}{2\sigma^2}\left\{ \mathrm{Re}\{\langle y(t), S_i(t)e^{j\theta}\rangle - \frac{\| S_i(t) e^{j\theta} \|^2}{2} \right\}}$$

Let $Z_i = \langle y(t), S_i(t)\rangle$

$$\langle y(t), S_i(t)\, e^{j\theta}\rangle = e^{-j\theta} \langle y(t), S_i(t)\rangle$$

$$\underbrace{\qquad\qquad}_{z_i}$$

$$z_i = |z_i| e^{j\phi_i}, \quad \phi_i = \angle z_i$$

$$\mathrm{Re}\,\langle y(t), s_i(t) e^{j\theta} \rangle = |z_i| \cos(\phi_i - \theta)$$

So $$L(y|H_i,\theta) = e^{\frac{1}{2\sigma^2}\left\{ |z_i| \cos(\phi_i-\theta) - \frac{\|s_i(t)\|^2}{2} \right\}}$$

Assume $\|s_1(t)\|^2 = \|s_2(t)\|^2 = \cdots = \|s_M(t)\|^2 = E_s$

$$L(y|H_i,\theta) = e^{\frac{1}{2\sigma^2}\left\{ |z_i| \cos(\phi_i-\theta) - \frac{E_s}{2} \right\}}$$

Now $$L(y|H_i) = \int_0^{2\pi} L(y|H_i,\theta)\, f(\theta)\, d\theta$$

$$\int^{2\pi} \quad |z_i| \cos(\phi_i - \theta)$$

$$= \frac{e^{-\frac{E_s}{4\sigma^2}}}{2\pi} \int_0^{2\pi} e^{\frac{1}{2\sigma^2}|z_i| \cos(\phi_i - \theta)} \, d\theta$$

$$= e^{-\frac{E_s}{4\sigma^2}} I_0\left(\frac{|z_i|}{2\sigma^2}\right)$$

increasing function

$\rightarrow$ $I_0(x)$ $\rightarrow$ is modified bessel function of $0^{th}$ kind

ML Rule:

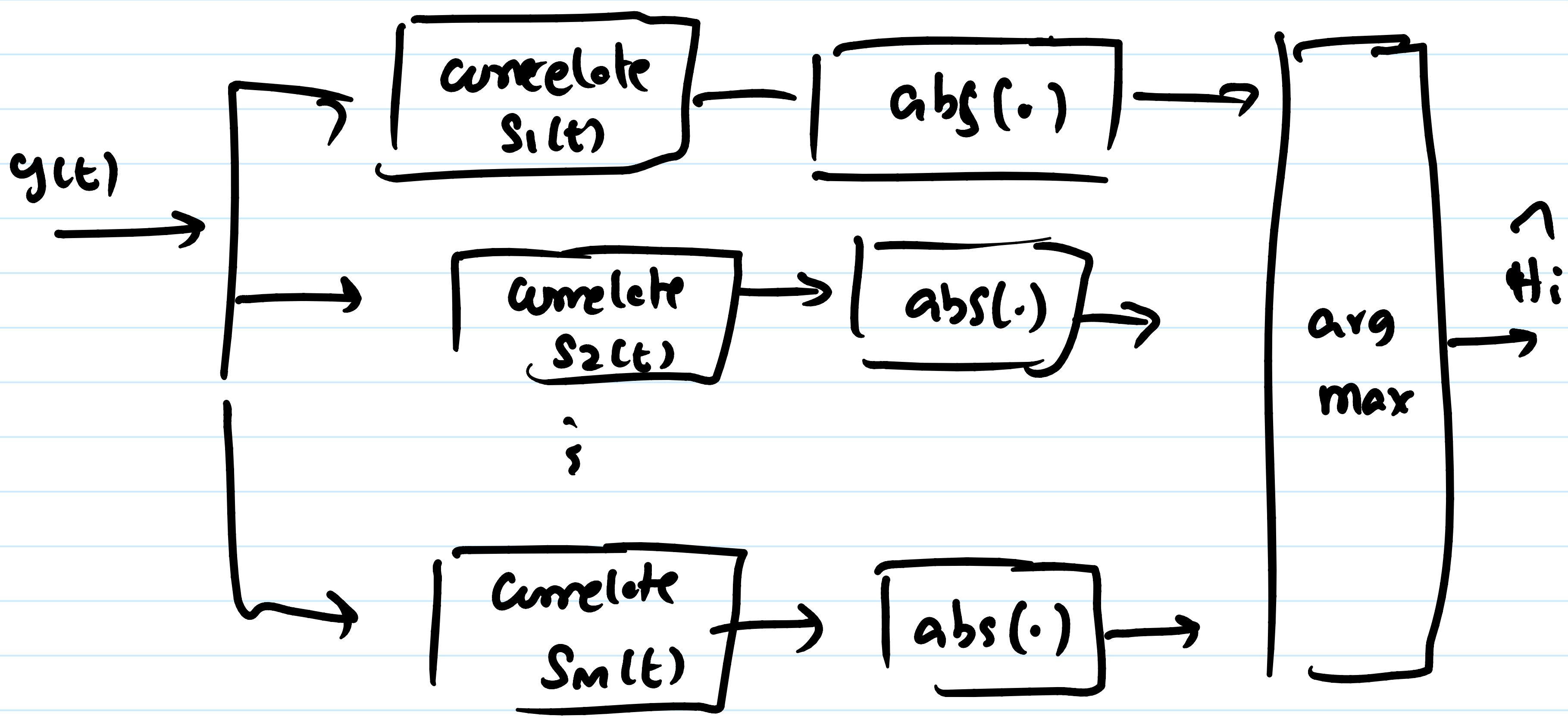
$$\delta_{ML}(y) = \underset{i}{\operatorname{argmax}} \; L(y | H_i)$$

$$= \underset{i}{\operatorname{arg\,max}} \; I_0\left(\frac{|z_i|}{2\sigma^2}\right)$$

$$\delta_{ML}(y) = \arg\max_i |z_i|$$

Recall $z_i = \langle y(t), s_i(t) \rangle$

Square-law detector (commonly known as)



## Differential Demodulation

(discrete-time)

$$y = s_i e^{j\theta} + w$$

$\theta \rightarrow$ random (& unknown)

$\theta \in [0, 2\pi]$

If $\theta$ remains constant over multiple symbol durations then information can be encoded using phase difference in adjacent symbols.

Recall Differential BPSK: $b[0] = 1$

$$b[n] = \begin{cases} b[n-1] & \text{if } n^{th} \text{ bit} = 0 \\ -b[n-1] & \text{if } n^{th} \text{ bit} = 1 \end{cases}$$

Differential QPSK $b(0) = e^{j\pi/4}$

$$b[n] = e^{j\frac{\pi}{2}q}\, b[n-1]$$

where $q$ takes one out of 4 values $(0, 1, 2, 3)$

For Differential PSK schemes,

$b(n)$ lies in unit circle for all $n$

ie) $|b(n)| = 1$

## Demodulator

$$y(n) = e^{j\theta} b(n) + w(n)$$

complex white noise

To demodulate $n^{th}$ symbol,

we need both $y(n-1)$ & $y(n)$

$$\begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} = e^{j\theta} \begin{bmatrix} b(n) \\ b(n-1) \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n-1) \end{bmatrix}$$

$$= e^{j\theta} b(n-1) \begin{bmatrix} \frac{b(n)}{b(n-1)} \\ \end{bmatrix} + \begin{bmatrix} w(n) \\ \end{bmatrix}$$

$$= \underbrace{e^{j\tilde{\theta}}}_{e} \; b(n-1) \begin{bmatrix} \frac{b(n)}{b(n-1)} \\ 1 \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n-1) \end{bmatrix}$$

$$a(n) = \frac{b(n)}{b(n-1)}$$

$$\underbrace{\begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix}}_{\underline{y}} = e^{j\tilde{\theta}} \underbrace{\begin{bmatrix} a(n) \\ 1 \end{bmatrix}}_{\underline{s}_a} + \underbrace{\begin{bmatrix} w(n) \\ w(n-1) \end{bmatrix}}_{\underline{w}}$$

$\tilde{\theta}$ is unknown

Note $a(n)$ also lies in unit circle

$\underline{s}_a \rightarrow$ Signal vector

It depends on symbol $a(n)$

$\|\underline{s}_a\|$ is constant for any value of $a(n)$

( Since $|a(n)| = 1$)

$\Downarrow$

Equal Energy Signalling

# Equal Energy Signalling

Optimal Non coherent demodulator is

$$\hat{a}(n) = \underset{a(n) \in \Lambda}{\arg\max} \left| \langle \underline{y}, \underline{s}_a \rangle \right|$$

demod output

$\Lambda \rightarrow$ set of all possible values for $a(n)$

Note : Diff. BPSK $\Lambda = \{+1, -1\}$

Diff QPSK $\Lambda = \{ e^{j\pi/2 \, q}, \ q = 0,1,2,3 \}$

$$\hat{a}(n) = \underset{a(n) \in \Lambda}{\arg\max} \left| \langle \underline{y}, \underline{s}_a \rangle \right|^2$$

Example:

Example.

Diff. BPSK $(a(n) = \pm 1)$

Optimal demod

$$|\langle \underline{y}, \underline{s}_{+1}\rangle|^2 \underset{-1}{\overset{+1}{\gtrless}} |\langle \underline{y}, \underline{s}_{-1}\rangle|^2$$

$$\underline{s}_{+1} = \begin{bmatrix} +1 \\ 1 \end{bmatrix} \qquad \underline{s}_{-1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Note: $\underline{s}_{+1}$, $\underline{s}_{-1}$ are orthogonal

$\underline{y}$ with some algebra

$$\boxed{\mathrm{Re}\left\{ y(n)\, y^*(n-1) \right\} \underset{-1}{\overset{+1}{\gtrless}} 0}$$

→ ML Rule for Diff. BPSK

In general for Diff. M PSK

$\Lambda$ → denote set of all values for $a(n)$

Optimal ML demod

$$\hat{a}(n) = \underset{a \in \Lambda}{\arg\max}\ \mathrm{Re}\left\{ y(n)\, y^*(n-1)\, a^* \right\}$$

x ——————————— x

# Performance Analysis for Noncoherent Demodulation

## Binary Equal Energy Signalling

$$H_0 : \quad y(t) = s_0(t)\, e^{j\theta} + w(t)$$

$$H_1 : \quad y(t) = s_1(t)\, e^{j\theta} + w(t)$$

Suff. Statistics

$$\|s_0(t)\|^2 = \|s_1(t)\|^2 = E_s = E_b$$

$$Z_0 = \langle y(t), s_0(t) \rangle$$

$$Z_1 = \langle y(t), s_1(t) \rangle$$

$$w(t) = w_c(t) + j\, w_s(t)$$

Complex WGN

$\cancel{Z_0} =$

Conditional Prob of Error given $H_0$

decision rule

$$|Z_0| \underset{H_1}{\overset{H_0}{\gtrless}} |Z_1|$$

$$P_{e|0} = \Pr\left\{ |Z_0| < |Z_1| \;\middle|\; H_0 \right\}$$

$\cancel{Z_0}$ Under $H_0$ : $y(t) = S_0(t)\, e^{j\theta} + w(t)$

$$Z_0 = \|S_0(t)\|^2 e^{j\theta} + \langle w(t), S_0(t) \rangle$$

$$= E_s\, e^{j\theta} + W_0$$

$$Z_1 = \langle S_0(t), S_1(t) \rangle e^{j\theta} + W_1$$

$$Z_1 = \langle S_0(t), S_1(t) \rangle e^{j\theta} + \underbrace{W_1}_{\langle w(t), S_1(t) \rangle}$$

Note

Let us define correlation coefficient

$-1 \le \rho \le 1$

$$\rho = \frac{\langle S_0(t), S_1(t) \rangle}{\| S_0(t) \| \, \| S_1(t) \|}$$

$$\rho E_s = \langle S_0(t), S_1(t) \rangle$$

$$Z_1 = \rho E_s e^{j\theta} + W_1$$

$$Z_0 = E_s e^{j\theta} + W_0$$

$$P_r \{ |Z_0| < |Z_1| \mid H_0 \}$$

Joint statistics of $w_1$ & $w_0$ can be got from properties of WGN

Probability computation involves integration of Gaussian pdfs over some (decision) region

For a general value of $\rho$,

we do not have closed form expression for $P(|z_0| < |z_1| \mid H_0)$

But we have asymptotic characterization

For $\rho = 0$, $|z_0|$ is Ricean distributed

$|z_1|$ is Rayleigh distributed

we have closed-form expression for $P_e$

**Theorem**

Binary Equal energy ($E_s$) signalling

$E_s$ = Avg. Symbol energy

For binary signalling,

we have $E_s = E_b$

$$P_e \text{ (non.coh. ML Rule)} \sim e^{-\left\{\frac{E_s}{2N_0}(1-|\rho|)\right\}}$$

$P_e$ (non. coh. ML Rule) $\sim e^{\,\cdot\,/2N_0\,(\cdots)}$

asymptotic as $\frac{E_s}{N_0} \to \infty$

when binary $\rho = 0$ ( orthogonal equal energy Signalling)

$$P_{e\,(\text{non coh})} = \frac{1}{2} e^{-E_s/2N_0}$$

equality

Recall <u>Coherent</u> binary Signalling

$$P_e = Q\left(\frac{\|s_1 - s_0\|}{2\sigma}\right)$$

we have

$$\|s_1 - s_0\|^2 = 2E_s\left[1 - \mathrm{Re}\{\rho\}\right]$$

$$\sigma^2 = \frac{N_0}{2} \qquad \rho = \frac{\langle s_0, s_1\rangle}{\|s_1\|\,\|s_0\|}$$

$$Q(x) \sim e^{-x^2/2} \quad \text{as} \quad x \to \infty$$

$$P_e(\mathrm{coh}) \sim e^{-\frac{E_s}{2N_0}\left[1 - \mathrm{Re}\{\rho\}\right]}$$

Remarks.

* Coherent comm. with $Re\{e\} < 0$ is better than non. coh. comm.

* For Coherent comm, best error performance is got when $Re\{e\} = -1$

$\Downarrow$

This happens for BPSK

$$S_0 = p(t)$$
$$S_1 = -p(t)$$

* For non-coh. comm, best error perf. is got

error perf. is got

when $|e| = 0$

$\Downarrow$

This happens when

we have orthogonal Signalling

$\times$ ———————— $\infty$

M-ary orthogonal equal energy Signalling

$$\|S_1\| = \|S_2\| = \dots = \|S_M\| = E_s$$

Union Bound

$$P_{e|1} \leq \sum_{i=2}^{M} P(S_1 \to S_i)$$

$\Downarrow$ Pairwise error Prob.

$$\downarrow$$

$$\frac{1}{2} e^{-E_s/2N_0}$$

$$P_{e|1} \le \frac{(M-1)}{2} e^{-\left(\frac{E_s}{2N_0}\right)}$$

Because of Symmetry

$$\text{Avg } P_e \le \frac{(M-1)}{2} e^{-\frac{E_s}{2N_0}}$$

$$E_b = \frac{E_s}{\log M}$$

$$P_{e\,(\text{non coh})} \leq \frac{(M-1)}{2} e^{-\frac{E_b \log M}{2 N_0}}$$

(equal energy)

We can establish that for M-ary orthogonal Signalling

$P_e(\text{non coh}) \rightarrow 0$ as $M \rightarrow \infty$

if $\frac{E_b}{N_0} > \ln 2$

$(-1.6\ \text{dB})$

&

Same limit as coherent Comm