Baseband Equivalent of Passband Filtering

Complex Envelope

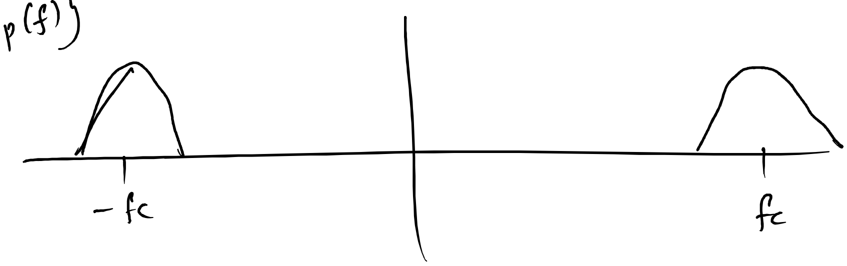
$$x(t) = x_c(t) + j\, x_s(t)$$

$x_c(t)$, $x_s(t)$ are real baseband signals

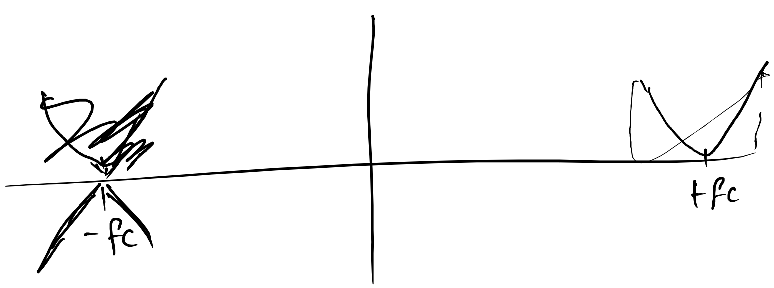
Passband Signal $x_p(t) = x_c(t) \cos 2\pi f_c t$
$\qquad\qquad - x_s(t) \sin 2\pi f_c t$

$X_p(f)$ be spectrum of $x_p(t)$

$Re\{X_p(f)\}$



$Im\{X_p(f)\}$



Whats is Spectrum of $x(t)$?

Define $X_p^+(f) = \{\ X_p(f)\ ;\ f \geq 0$

$0 \quad ; \quad$ else

Complex baseband Spectrum

$$X(f) = 2 X_p^+(f+f_c)$$

$$x_p(t) \longrightarrow \boxed{\begin{matrix} LTI \\ g(t) \end{matrix}} \longrightarrow y_p(t)$$

x_p, y_p are passband Signals

$g(t) \to$ impulse ~~of~~ response of filter

$x(t) \to$ baseband equivalent of $x_p(t)$

$y(t) \to$ " " " $y_p(t)$

Given x_p, y_p, & $g(t)$

how to find relation between $x(t)$ & $y(t)$

$X_p(f) \to$ restricted in band to

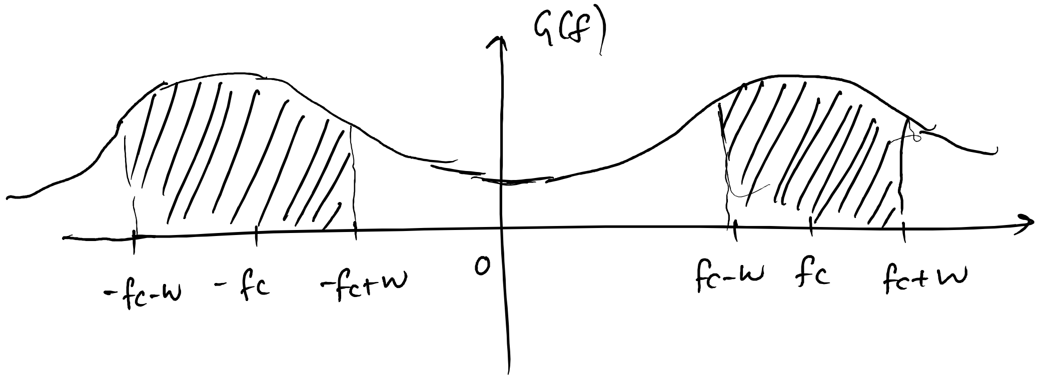
$f_c - W$ to $f_c + W$

(with symmetry on -ve side)

Since System is LTI,

$Y_p(f)$ is also limited to $f_c - W$ to $f_c + W$
in band

$g(t) \rightarrow$ impulse response
of system
$G(f) \rightarrow$ frequency response



Define $$H_p(f) = \begin{cases} G(f) \ ; & |f \pm f_c| \leq W \\ 0 \ ; & \text{else} \end{cases}$$

It is spectrum of a passband signal $h_p(t)$

$$h_p(t) \xleftrightarrow{F} H_p(f)$$

Clearly, convolution property of Fourier Transform gives

$$Y_p(f) = X_p(f) \, H_p(f)$$

Equivalently

$$y_p(t) = x_p(t) * h_p(t)$$

Say we have the following baseband representations

$$\left.\begin{array}{l} y_p(t) \longrightarrow y(t) \\ x_p(t) \longrightarrow x(t) \\ h_p(t) \longrightarrow h(t) \end{array}\right\} \text{ baseband equivalents}$$

Claim : $\quad y(t) = \frac{1}{2} x(t) * h(t)$

<u>Proof:</u>

We have

$$Y_p(f) = X_p(f) \; H_p(f)$$

Now, $\quad Y(f) = 2 \, Y_p^+(f+f_c)$

$$X(f) = 2 \, X_p^+(f+f_c)$$

$$H(f) = 2 \, H_p^+(f+f_c)$$

$$Y(f) = 2 \, Y_p^+(f+f_c)$$

$$= 2 \, X_p^+(f+f_c) \cdot H_p^+(f+f_c)$$

$$= \frac{1}{2} X(f) \cdot H(f)$$

baseband equivalent filter.

$\Rightarrow \quad y(t) = 1 \, (x(t) * h(t))$

× ———————— ×

Other Equivalents between baseband & Passband signals

$$x(t) = x_c(t) + j\,x_s(t)$$

$$x_p(t) = \cancel{x_c(t)\, e^{j2\pi f_c t}}$$

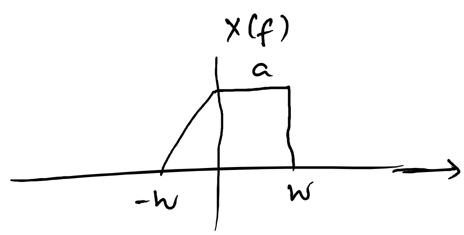
$$= x_c(t) \cos 2\pi f_c t - x_s(t) \sin 2\pi f_c t$$

Claim : $$\int_{-\infty}^{\infty} |x_p(t)|^2 dt = \frac{1}{2} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

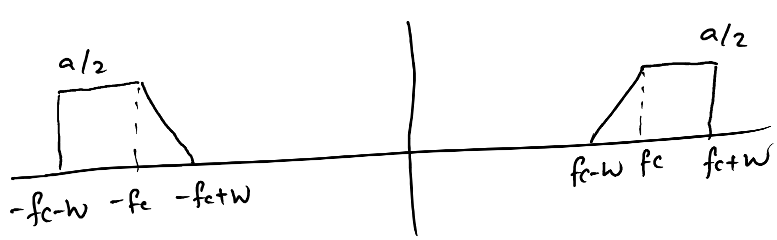
Proof: Time domain integration (do it yourself)

$X_p(f) \rightarrow$ F.T. of $x_p(t)$

$X(f) \rightarrow$ F.T. of $x(t)$



$X_p(f)$



$$\int_{-\infty}^{\infty} |\ \ |^2 \ldots = \int^{-f_c+W} |X_p(f)|^2 df$$

Clearly $\int_{-\infty}^{\infty} |X_p(f)|^2 df = \int_{-f_c-W}^{-f_c+W} |X_p(f)|^2 df + \int_{f_c-W}^{f_c+W} |X_p(f)|^2 df$

$$= \frac{1}{4} \int_{-W}^{W} |X(f)|^2 df + \frac{1}{4} \int_{-W}^{W} |X(f)|^2 df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |X(f)|^2 df$$

$x_p(t)$, $y_p(t)$ are two different passband signals

$\downarrow \quad \downarrow$

$x(t)$ $y(t)$ are baseband equivalent representations

Claim: $\langle x_p(t), y_p(t)\rangle = \int_{-\infty}^{\infty} x_p(t)\, y_p(t)\, dt$

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

We have $\langle x_p(t), y_p(t)\rangle = \frac{1}{2} \operatorname{Re}\{\langle x(t), y(t)\rangle\}$

Proof: Verify yourself

$\langle x_p(t), y_p(t)\rangle$

$$= \frac{1}{2}\left\{\langle x_c(t), y_c(t)\rangle + \langle x_s(t), y_s(t)\rangle\right\}$$

$\langle x(t), y(t) \rangle$

$$= \langle x_c(t), y_c(t) \rangle + \langle x_s(t), y_s(t) \rangle + j \{ \langle x_s(t), y_c(t) \rangle - \langle x_c(t), y_s(t) \rangle \}$$

$\times$ —— $\times$