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Linear Equalizers

Suboptimal but low complexity techniques to recover the stream of symbols in ISI channels

Let $g(t)$ be pulse shape for modulation

$g_c(t)$ be channel impulse response

$p(t) = g(t) * g_c(t)$

effective pulse shape

Recall, to implement MLSE, we ~~Comp~~ build matched filter with impulse response $p^*(-t)$

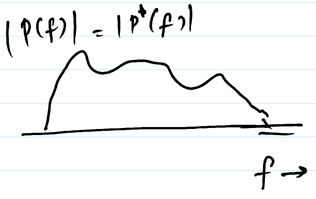
(Spectrum)

Now, let $p(t) \longleftrightarrow P(f)$

$p^*(-t) \longleftrightarrow P^*(f)$

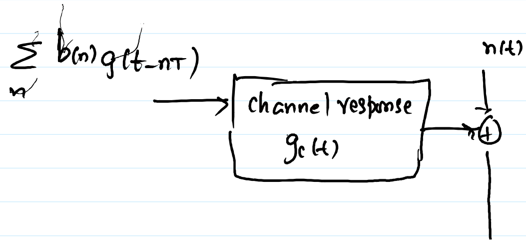
Matched filter should have frequency response $P^*(f)$.

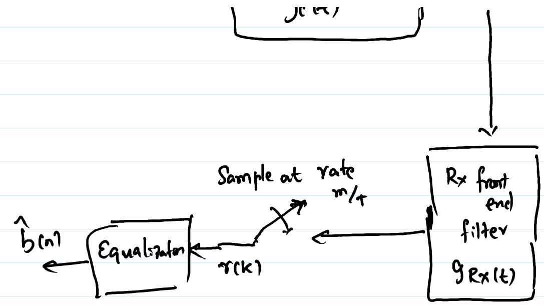
Building analog filter with given arbitrary response is not easy.



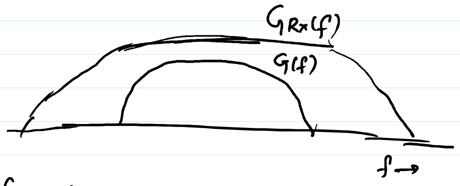
Typical Receiver implementation

use a generic/simple filter at the receiver ~~front~~ end.





① Since original pulse shape $g(t)$ is typically band limited we can predesign $g_{Rx}(t)$



$G_{Rx}(f)$ → allows the entire Signal spectrum to pass through.

→ Also has long transition band for ease in implementation

② Sampling rate $\frac{m}{T}$, m is positive integer

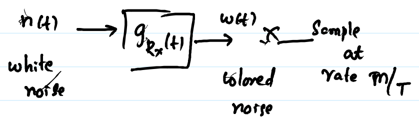
- $m=1$ Symbol spaced sampling
- $m>1$ fractional spaced sampling

Impact of $g_{Rx}(t)$ on the Signal and noise

Signal : $\sum b(n)\, g(t-nT) * g_c(t) * g_R(t)$

$\downarrow$ Sample at rate $\frac{m}{T}$

Noise :



First, let us look at noise statistics

$n(t)$ → complex WGN with variance $2\sigma^2$

$$w(t) = n(t) * g_{Rx}(t)$$

$w(t)$ is colored Gaussian noise

with auto correlation function

$$R_w(\tau) = E\{w(t)\, w^*(t-\tau)\}$$

$$= 2\sigma^2 \int_{-\infty}^{\infty} g_{Rx}(t)\, g^*_{Rx}(t-\tau)\, dt$$

Let $w(k) = w(t)$ sampled at $t = k\frac{T}{m}$

$\downarrow$

discrete time noise sequence

Now, ACF of discrete time noise sequence

$$E\{w(k)\, w^*(k-l)\} = R_w(lT/m)$$

from CT ACF, we can get DT ACF.

Now, let us look at $\underline{\text{signal component}}$

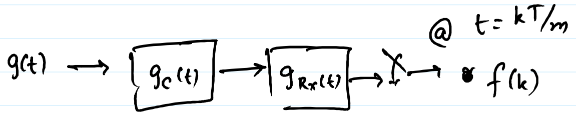
$$\sum_n b(n)\, g(t-nT) * g_c(t) * g_{Rx}(t) \rightarrow \text{Sampled at } t = kT/m$$

$\downarrow$

$r(k)$

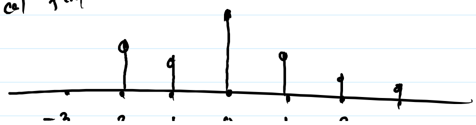
what is the relation between $b(n)$ & $r(k)$?

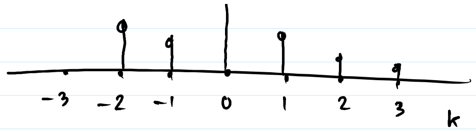
Say, we send a single pulse $g(t)$



$$f(k) = g(t) * g_c(t) * g_{Rx}(t) \Big|_{t = kT/m}$$

Typical $f(k)$





$f(k) \rightarrow$ usually a finite duration sequence

Input $\qquad\qquad$ Net Output

$$g(t) \longrightarrow f(k) \qquad k \in \mathbb{Z}$$

$$b(0)\, g(t) \xrightarrow[\text{linearity}]{} b(0)\, f(k)$$

$$g(t-T) \xrightarrow[\text{Time invariance}]{} f(k-m)$$

Shift by symbol duration

$\downarrow$ Shift by symbol duration (m samples in each symbol duration)

$$b(n)\, g(t-nT) \longrightarrow b(n)\, f(k-mn)$$

$$\sum b(n)\, g(t-nT) \longrightarrow \sum_n b(n)\, f(k-mn)$$

$$\| \\ r(k)$$

In the presence of noise, we have the following DT model

$$\boxed{r(k) = \sum_n b(n)\, f(k-nm) + w(k)}$$

$b(n) \rightarrow$ input symbols

$r(k) \rightarrow$ received samples @ rate m/T

$f(k) \rightarrow$ DT response for single pulse $g(t)$

$w(k) \rightarrow$ colored noise samples

As long as filter $g_{Rx}(t)$ and

sampling rate m/T are

chosen appropriately there is

no loss / very little loss of information

from CT to above DT model)

Also, since $f(k)$ is usually of finite duration,

only few input samples around

$\{b(k/m)\}$ will have

contribution to $r(k)$

Or, symbols around $\{b(n)\}$ will have

contribution to $r(nm)$

x —————————— x

We consider a specific symbol $\underline{b(n)}$

We want to estimate the value of $b(n)$

using L observations which have

significant contribution from $b(n)$

L samples centered around $k = nm$

Collect them as a vector $\underline{r}(n)$

$\underline{r}(n) \rightarrow$ will have signal contributions

from say

$$b(n-k_1), \ldots, b(n), \ldots, b(n+k_2)$$

We can write

$$\underline{r}(n) = \underline{u}_0\, b(n) + \underline{u}_{-1}\, b(n-1) + \ldots + \underline{u}_{-k_1} b(n-k_1)$$
$$+ \underline{u}_1\, b(n+1) + \ldots + \underline{u}_{k_2}\, b(n+k_2)$$
$$+ \underline{w}(n)$$

$\underline{u}_i$ vectors are constructed using our DT model

(they contain entries from $\{f(u)\}$)

$\underline{w}(n)$ → Colored noise vector with Covariance matrix C_w

Now,

$$\underline{r}(n) = \underbrace{[\underline{u}_{-k_1} \cdots \underline{u}_0 \cdots \underline{u}_{k_2}]}_{U} \underbrace{\begin{bmatrix} b(n-k_1) \\ \vdots \\ b(n) \\ \vdots \\ b(n+k_2) \end{bmatrix}}_{\underline{b}(n)} + \underline{w}(n)$$

$$\boxed{\begin{array}{cccccc} \underline{r}(n) & = & U & \underline{b}(n) & + & \underline{w}(n) \\ L\times 1 & & L\times k & k\times 1 & & L\times 1 \end{array}}$$

Vector model for recovering a specific symbol $b(n)$

Note: • Typically $L > k$ when $m > 1$

- To get vector model for $b(n+1)$

  we shift $\underline{b}(n)$ by 1 unit

  $\underline{r}(n)$ by m units

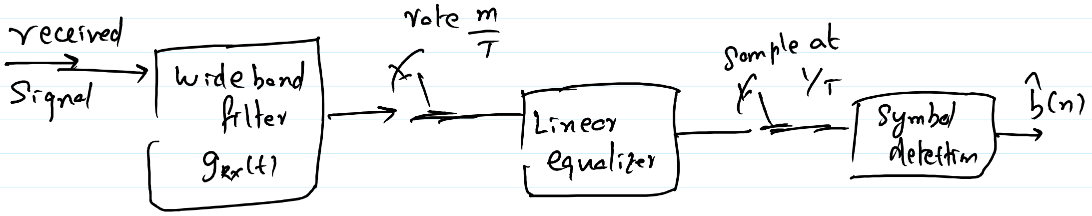
  $\underline{U}$ & C_w remain same

We consider two linear equalizers

① Zero forcing

② MMSE

Overall architecture



Linear Equalizer :

Using $\underline{r}(n)$, we want to recover symbol $b(n)$

$\underset{L\times 1 \text{ vector}}{\underline{r}(n)}$ $\quad$ $\underset{\text{single symbol}}{b(n)}$

Let $\underline{c}$ be a $L\times 1$ vector

We compute decision statistic

$$\begin{aligned} Z(n) &= \langle \underline{r}(n), \underline{c} \rangle \\ &= \underline{c}^{*}\, \underline{r}(n) \\ &= \underline{c}^{*} \underline{u}_0\, b(n) + \sum_{\substack{i \neq 0 \\ i=-k_1}}^{k_2} \underline{c}^{*} \underline{u}_i\, b(n-i) \end{aligned}$$

$$\begin{matrix} i \neq 0 \\ i = -k_1 \end{matrix} \qquad - -$$

$$+ \quad \underline{c}^* \underline{w}$$

Signal component : $\underline{c}^* \underline{u}_0$

ISI components : $\underline{c}^* \underline{u}_i \quad i \neq 0$

noise component : $\underline{c}^* \underline{w}$

To make reliable decision on $b(n)$ using $z(n)$, we need

Signal component to be high

ISI & noise component to be Small

Goal : Design $\underline{c}$ suitably.

① Zero Forcing Equalizer.

Criterion :

$$\left.\begin{aligned} \text{Want} \quad & \underline{c}^* \underline{u}_0 = 1 \\ & \underline{c}^* \underline{u}_i = 0 \text{ for } i \neq 0 \\ & \qquad\qquad i = -k_1 \text{ to } k_2 \end{aligned}\right\} (*)$$

Or equivalently

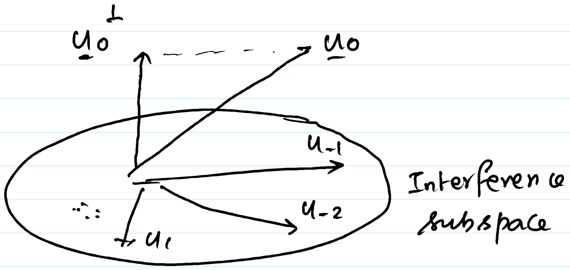
$$\underline{c}^H U = [0, \ldots, 0, \underset{\substack{\downarrow \\ \text{at location of } b(n)}}{1}, 0, \ldots, 0]$$

or

$$U^H \underline{c} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\underline{e}}$$

$$\ast \quad U^H \underline{c} = \underline{e}$$

Geometric View



Solution for (*) exist if & only if

$\underline{u}_0$ is linearly independent of

$\{ \underline{u}_i \quad i \neq 0,\ i = -k_1 \text{ to } k_2 \}$

In this case, we have

$$\boxed{\underline{c} = U(U^H U)^{-1} \underline{e}}$$

pseudo inversion.

Now

$$\underline{u}_0 = \underbrace{\underline{u}_0^{I}}_{\substack{\text{component} \\ \text{of } \underline{u}_0 \\ \text{along} \\ \text{interference} \\ \text{subspace}}} + \underbrace{\underline{u}_0^{\perp}}_{\substack{\text{component of } \underline{u}_0 \\ \text{orthogonal to} \\ \text{interference} \\ \text{subspace.}}}$$

Note: $\underline{c}^{*} \underline{u}_0^{I} = 0$

After ZF equalization,

Only orthogonal component $\underline{u}_0^{\perp}$ contributes to signal

ZF output SNR $\propto \| \underline{u}_0^{\perp} \|^2$

Noise Enhancement factor $= \frac{1}{{}^{2}}$

noise enhancement factor = $\frac{}{\|\underline{u}_0^{\perp}\|^2}$

Intuitively

ZF nulls out* ISI completely

but enhances noise

(by reduction of signal component)

* ______________

MMSE Equalization:

- Strikes a balance between removal of ISI & noise enhancement
- Allows for some residual ISI but overall SINR is better than ZF
- MMSE criterion

choose $\underline{c}$ such that

$$E\{|b(n) - \underline{c}^{*}\underline{r}(n)|^2\} \text{ is minimum}$$

- optimal $\underline{c}$ is given by

$$\underline{c}_{MMSE} = Q^{-1}\underline{p}$$

where

$$Q = E\{\underline{r}(n)\,\underline{r}^*(n)\}$$

$$\underline{p} = E\{\underline{r}(n)\,b^*(n)\}$$

- Suppose ~~input~~ symbols ~~are~~ uncorrelated (~~zero~~ mean)

$$E\{b(n)\,b^*(m)\} = \begin{cases} 0 & \text{if } m \neq n \\ \sigma_b^2 & \text{if } m = n \end{cases}$$

then

$$Q = \sigma_b^2\, U U^* + C_w$$

$$\underline{p} = \sigma_b^2\, \underline{u}_0$$

- SIR at output

$$= \frac{\sigma_b^2\, |\underline{c}^* \underline{u}_0|^2}{\sigma_b^2 \sum_{i \neq 0} |\underline{c}^* \underline{u}_i|^2 + \underline{c}^* C_w \underline{c}}$$

- MMSE equalizer maximises SIR among all linear equalizers

- When Noise is small $\sigma^2 \approx 0$ then MMSE & ZF coincide.

x ———————— x

Decision Feedback Equalization (DFE)

Recall

$$\underline{r}(n) = \underline{u}_0 \, b(n) + \underline{w}(n)$$

$$+ \sum_{i=1}^{k_1} b(n-i) \, \underline{u}_{-i}$$ → ISI from past symbols

$$+ \sum_{i=1}^{k_2} b(n+i) \, \underline{u}_{i}$$ → ISI from future symbols

DFE Concept

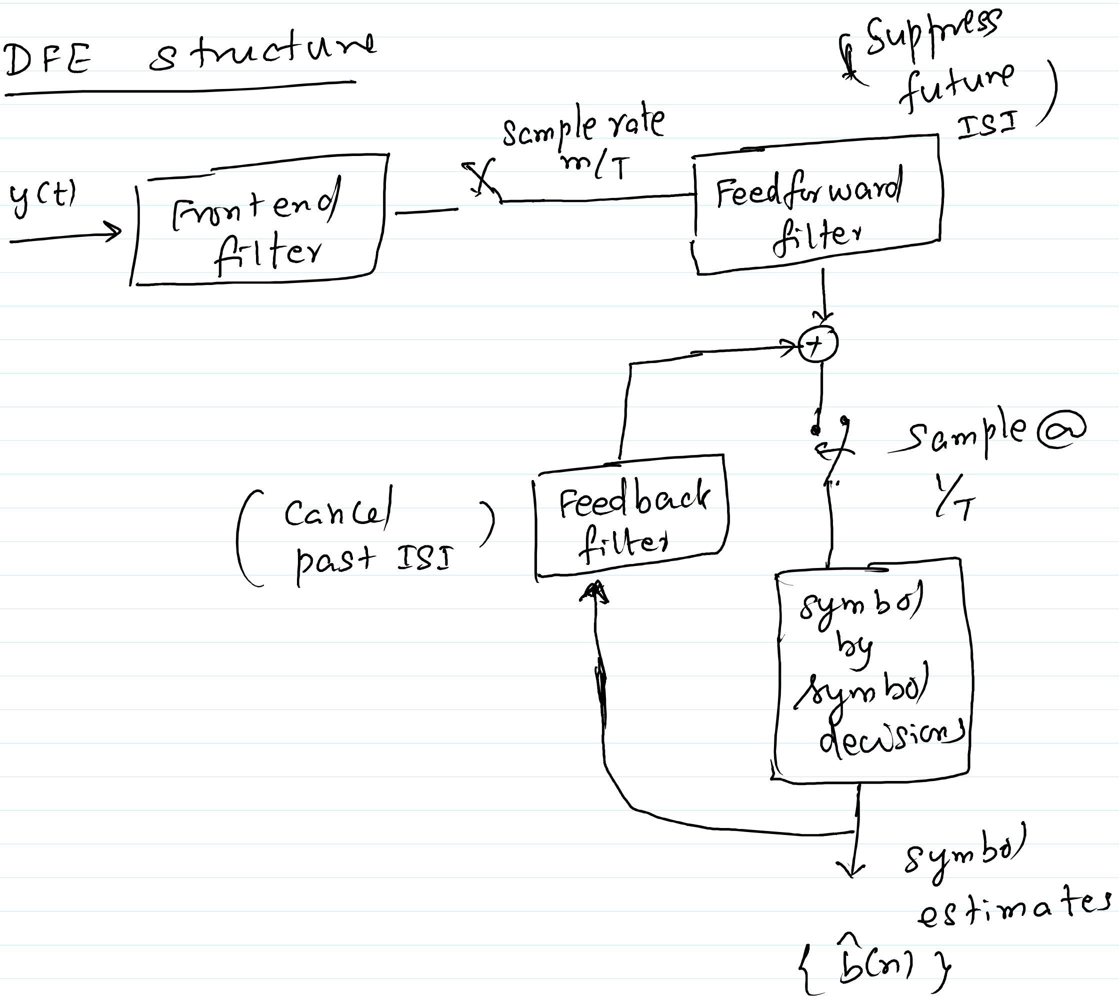
- Since past symbols $b(n-i), i>0$ have already been demodulated we can use those symbol estimates to cancel the ISI from past symbols

- Since past ISI is removed (under error free demodulation) linear equalizer has to deal only with ISI from future symbols

- reduced dimension of ISI helps in improving the performance of linear equalizer

- ONE Problem: Error propagation

DFE structure



Math details

$\underline{c}_{FF} \rightarrow$ feed forward equalizer

Consider

$$\underline{c}_{FF}^{*}\,\underline{r}(n) = \underline{c}_{FF}^{*}\,\underline{u}_0\, b(n) + \underline{c}_{FF}^{*}\,\underline{w}(n) + \sum_{i=1}^{k_2} \underline{c}_{FF}^{*}\,\underline{u}_i\, b(n+i) + \underbrace{\sum_{i=1}^{k_1} \underline{c}_{FF}^{*}\,\underline{u}_{(-i)}\, b(n-i)}_{\text{Past ISI after FF equalizer}}$$

Feed back filter need to cancel this

Let $\hat{b}(n-i)$, $i>0$ denote estimate of past symbols

Set $$c_{FB}(i) = -\underline{c}_{FF}^{*}\,\underline{u}_{-i}$$

Decision statistic of DFE

$$Z_{DFE}(n) = \underline{c}_{FF}^{*}\,\underline{r}(n) + \sum_{i=1}^{k_1} c_{FB}[i]\,\hat{b}(n-i)$$

$$= \underline{c}_{FF}^{*} \underline{u}_0 \, b(n) + \underline{c}_{FF}^{*} \underline{w}(n)$$

$$+ \sum_{i=1}^{k_2} \underline{c}_{FF}^{*} \underline{u}_i \, b(n+i)$$

$$+ \sum_{i=1}^{k_1} c_{FF}^{*} \underline{u}_{-i} \left[b(n-i) - \hat{b}(n-i) \right]$$

- If feedback is perfect, contributions from past symbols is cancelled correctly

- Feed forword filter can be chosen based on ZF or MMSE criterion

We get ZF-DFE or MMSE-DFE