23 September 2016 10:58

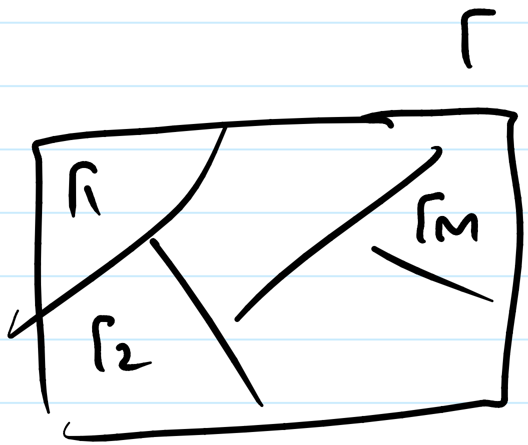
Observation $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}$ → random vector

$\underline{y}$ → one particular realization (outcome)

For each $\underline{y} \in \mathbb{R}^N$, we have to specify what the decision is.

decision rule $\delta(\underline{y}) = \begin{cases} H_i & \text{if } \underline{y} \in \Gamma_i \end{cases}$

observation space $\Gamma = \mathbb{R}^N$



We design decision regions $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$ to minimize

Average probability of error

Conditional Probability of Error

$P_{e|i} \rightarrow$ Cond. Prob. of Error Given that H_i is true hypothesis

$$P_{e|i} = P_r\{\delta(\underline{y}) \neq H_i \mid H_i \text{ is true}\}$$

$$= P_r\{\underline{y} \notin \Gamma_i \mid H_i\}$$

$$= 1 - \underbrace{P_r\{\underline{y} \in \Gamma_i \mid H_i\}}_{P_{c|i}}$$

$P_{c|i} \rightarrow$ Cond. prob. of making correct decision given H_i is true

Average Prob. of Error

$$P_e \text{ (unconditional)} = \sum_{i=1}^{M} \pi_i P_{e|i}$$

Prior prob. $\pi_i \rightarrow Pr\{H_i \text{ is true}\}$

$$P_c \text{ (unconditional)} = 1 - P_e$$

Avg. Prob. of correct decision

$$= 1 - \sum_{i=1}^{M} \pi_i P_{e|i}$$

Want to minimize P_e by choosing $\Gamma_1, \Gamma_2, \ldots \Gamma_M$ appropriately

Minimizing P_e $\Leftrightarrow$ (equivalent) maximize P_c

$$P_c = 1 - P_e = \sum_{i=1}^{M} \pi_i \, P_{c|i}$$

$$= \sum_{i=1}^{M} \pi_i \, \Pr\{\underline{y} \in \Gamma_i \mid H_i\}$$

$$= \sum_{i=1}^{M} \pi_i \int_{\Gamma_i} f(\underline{y} \mid H_i) \, d\underline{y}$$

$$P_c = \sum_{i=1}^{M} \underbrace{\int_{\Gamma_i} \pi_i \, f(\underline{y} \mid H_i) \, d\underline{y}}_{\text{Want to maximize}}$$

$\Gamma_1, \Gamma_2, \ldots, \Gamma_M$ are disjoint regions

π_i, $f(\underline{y} \mid H_i)$ are non-negative

Each $\underline{y}$ in $\mathbb{R}^n$ has to be assigned to one of regions $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$

Integrand is $\pi_i f(\underline{y} \mid H_i)$

For a given $\underline{y}$, compute

$$\pi_i \, f(\underline{y} \mid H_i) \quad \text{for } i = 1 \text{ to } M$$

assign $\underline{y}$ to Γ_j if

$$j = \underset{1 \leq i \leq M}{\arg\max} \; \pi_i \, f(\underline{y} \mid H_i)$$

$$\delta(\underline{y}) = \left\{ H_j \quad \text{if } j = \underset{1 \leq i \leq M}{\arg\max} \; \pi_i \, f(\underline{y} \mid H_i) \right.$$

Min. Prob. of Error Rule (MPE)

$$\delta_{MPE}(\underline{y}) = \underset{1 \le i \le M}{\arg\max} \; \pi_i f(\underline{y} | H_i)$$

log(·) is monotonic increasing

$$= \underset{1 \le i \le M}{\arg\max} \; \log \pi_i + \log f(\underline{y} | H_i)$$

MAP Rule

Prior Probabilities $\pi_i = \Pr\{H_i \text{ is true}\}$

Posterior Probability (after we get observation $\underline{y}$)

||

$\Pr\{H_i \mid \underline{y}\}$

Maximum Aposteriori Prob. Rule (MAP)

$$\delta_{MAP}(\underline{y}) = \underset{1 \le i \le M}{\operatorname{argmax}} \Pr\{H_i \mid \underline{y}\}$$

Theorem $\quad \delta_{MAP}(\underline{y}) = \delta_{MPE}(\underline{y})$

MAP Rule is equivalent to Min. Prob. of Error Rule

(Bayes Rule) $\quad P(H_i \mid \underline{y}) = \dfrac{f(\underline{y} \mid H_i) \Pr\{H_i\}}{f(\underline{y})}$

total Prob. theorem $\quad f(\underline{y}) = \sum_{i=1}^{M} \pi_i \, f(\underline{y} \mid H_i)$

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Prob. theorem $\quad i=1$

$\downarrow$

unconditional pdf of $\underline{y}$

$$\delta_{MAP}(\underline{y}) = \arg\max_{i} \Pr\{H_i \mid \underline{y}\}$$

$$= \arg\max_{i} \frac{f(\underline{y}|H_i)\,\pi_i}{f(\underline{y})}$$

$f(\underline{y})$ does not depend on i

$$= \arg\max_{i} \pi_i\, f(\underline{y}|H_i)$$

$$= \delta_{MPE}(\underline{y})$$

Maximum Likelihood Rule (ML)

$$\delta_{ML}(\underline{y}) = \arg\max_{1 \le i \le M} f(\underline{y}|H_i)$$

$$\delta_{ML}(\underline{y}) = \arg\max_{1 \le i \le M} \ldots$$

If $\pi_1 = \pi_2 = \cdots = \pi_M$, then

$$\delta_{ML}(\underline{y}) = \delta_{MAP}(\underline{y})$$

Binary Hypothesis Testing

$\underline{M=2}$ Two Hypotheses H_0, H_1

$$\delta_{MPE}(y) = \begin{cases} H_1 & \text{if } \pi_1 f(y|H_1) > \pi_0 f(y|H_0) \\ H_0 & \text{if } \pi_0 f(y|H_0) > \pi_1 f(y|H_1) \\ H_1 \text{ or } H_0 & \text{if } \pi_0 f(y|H_0) = \pi_1 f(y|H_1) \end{cases}$$

equivalently

- equivalently

$$\pi_1 f(\underline{y}|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 f(\underline{y}|H_0)$$

Likelihood Ratio

$$\frac{f(\underline{y}|H_1)}{f(\underline{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0}{\pi_1}$$

log likelihood Ratio

$$\log \frac{f(\underline{y}|H_1)}{f(\underline{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \log(\pi_0/\pi_1)$$