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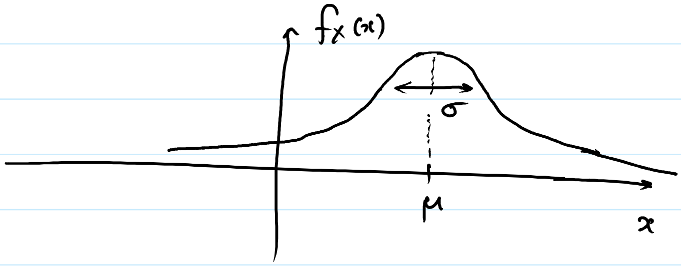
Gaussian Basics

X is a Gaussian random variable with mean μ and variance σ^2

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Shorthand Notation: $X \sim N(\mu, \sigma^2)$

Note: Suppose $X \sim N(\mu, \sigma^2)$

Let $Y = aX + b$

a, b constants

Then $Y \sim N(a\mu + b, a^2\sigma^2)$

Gaussian pdf is completely characterized by its mean & variance.

Definition of Q function

Suppose $X \sim N(0, 1)$ ⇘ standard normal pdf

(ccdf) Complementary cdf of X

$$P(X > x) = \int_x^\infty f_X(t)\,dt$$

∞ $-t^2/_2$

$$= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

definition of Q

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$= 1 - P(X \geq x) = \text{ccdf}$$

- $Q(x)$ is a decreasing function of x
- $Q(x) \leq \frac{1}{2} e^{-x^2/2}$

Suppose $Y \sim N(\mu, \sigma^2)$

Let $Z = \frac{Y-\mu}{\sigma}$

Note that $Z \sim N(0,1)$

Now, $P(Y > a)$

$$= P\left(\frac{Y-\mu}{\sigma} > \frac{a-\mu}{\sigma} \right)$$

$$= P\left(Z > \frac{a-\mu}{\sigma} \right)$$

$$= Q\left(\frac{a-\mu}{\sigma} \right)$$

<u>Central limit theorem</u> (CLT) (Significance of Gaussian)

Suppose $X_1, X_2, \ldots, X_N, \ldots$ be a sequence of i.i.d. random variables

(independent and identically distributed)

with mean μ and variance σ^2

Consider the "average"

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_i - \mu}{\sigma}$$

According to C.L.T.

S_N converges in distribution to $N(0,1)$

that is $\lim_{N \to \infty} P\{S_N > x\} = Q(x)$

x ———— x

Noise in communication system

comes from various (independent) sources

- thermal noise
- quantization noise
- shot noise
- noise in physical medium

Effective noise which is sum of

the noise from various sources

can be approximated with Gaussian statistics

x ———— x

Gaussian Random Vector

$$\underset{P \times 1}{\underline{X}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_P \end{bmatrix}$$

$x_1, x_2, \ldots, x_p$ are called jointly Gaussian random variables or

$\underline{x}$ is joint Gaussian random vector

if $a_1 x_1 + a_2 x_2 + \cdots + a_p x_p$ is a Gaussian random variable

for any real values of $a_1, a_2, \ldots, a_p$

Recall mean vector

$$\underline{\mu}_x = \begin{bmatrix} E(x_1) \\ \vdots \\ E(x_p) \end{bmatrix}$$

Covariance Matrix

$$C_x = E\left\{ (\underline{x} - \underline{\mu}_x)(\underline{x} - \underline{\mu}_x)^T \right\}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ \vdots & & & \\ c_{p1} & & \cdots & c_{pp} \end{bmatrix}$$

let $\mu_i = E(x_i)$

$$c_{ij} = E\left\{ (x_i - \mu_i)(x_j - \mu_j) \right\}$$

$$= \mathrm{Cov}(x_i, x_j)$$

$$c_{ii} = \mathrm{Var}(x_i) = \sigma_i^2$$

Note $x_i \sim N(\mu_i, \sigma_i^2)$

Joint pdf of $\underline{x}$ is completely characterized by $\underline{\mu}_x$ & C_x

$$f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |C_x|^{1/2}} e^{-\frac{1}{2} (\underline{x} - \underline{\mu}_x)^T C_x^{-1} (\underline{x} - \underline{\mu}_x)}$$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}\,|C_x|^{1/2}}\, e$$

$\searrow$ determinant

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

[Assume C_x is invertible]

Consider case: $X_1, X_2, \ldots, X_p$ are uncorrelated

$$\mathrm{Cov}(X_i, X_j) = 0$$

$$C_x = \begin{bmatrix} c_{11} & & & 0 \\ & c_{22} & & \\ & & \ddots & \\ 0 & & & c_{pp} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_p^2 \end{bmatrix}$$

$$\sigma_i^2 = \mathrm{Var}(X_i)$$

If X_i's are uncorrelated,
covariance matrix is diagonal

Let us find joint pdf for this case

Note $$C_x^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & & 0 \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sigma_p^2} \end{bmatrix}$$

determinant

$$|C_x| = \prod_{i=1}^{p} \sigma_i^2$$

$$-\frac{1}{2}(\underline{x}-\underline{\mu})^T C_x^{-1} (\underline{x}-\underline{\mu})$$

$$f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^{p} \sigma_i} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})' C_x^{-1} (\underline{x}-\underline{\mu})}$$

$$(\underline{x}-\underline{\mu})^T C_x^{-1} (\underline{x}-\underline{\mu})$$

$$= \left[(x_1-\mu_1) \quad (x_2-\mu_2) \ldots (x_p-\mu_p)\right]$$

$$\times \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \frac{1}{\sigma_2^2} & \\ 0 & & \ddots \quad \frac{1}{\sigma_p^2} \end{bmatrix}$$

$$\times \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_p-\mu_p \end{bmatrix}$$

$$= \sum_{i=1}^{p} \frac{(x_i-\mu_i)^2}{\sigma_i^2}$$

$$f_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^{p} \sigma_i} e^{-\frac{1}{2}\sum_{i=1}^{p} \frac{(x_i-\mu_i)^2}{\sigma_i^2}}$$

$$= \prod_{i=1}^{p} \frac{1}{(2\pi)^{1/2}\sigma_i} e^{-\frac{1}{2}\frac{(x_i-\mu_i)^2}{\sigma_i^2}}$$

$$= \prod_{i=1}^{p} f_{x_i}(x_i)$$

$i=1$

$\Downarrow$

joint f_X pdf = product of marginal pdfs

Hence if x_i's are jointly Gaussian and x_i's are uncorrelated then x_i's are <u>independent</u>.

- Any linear transformation of Gaussian random vector is also a Gaussian random vector.

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \qquad \underline{X} \sim N(\underline{\mu}_x, C_x)$$

Consider $\underset{M\times 1}{\underline{Y}} = \underset{M\times p}{A}\,\underline{X} + \underset{M\times 1}{\underline{b}}$

(A, $\underline{b}$ are constant matrix/vector)

We have $\underline{Y}$ is also a Gaussian vector

$$\underline{Y} \sim N(\underbrace{A\underline{\mu}_x + \underline{b}}, \underbrace{AC_xA^T})$$

μ_r $\quad$ C_r

x ———————— x

Gaussian Random Process

$\{X(t)\}$ is called a Gaussian Random Process if the samples $\{X(t_1), X(t_2), \ldots, X(t_N)\}$ are jointly Gaussian for any sampling instants $\{t_1, t_2, \ldots, t_N\}$ and N is any integer

Recall Mean function

$\mu(t) = E\{X(t)\}$

Auto Correlation function ACF

$$R_X(t;\, t-\tau) = E\{X(t)X(t-\tau)\}$$

From $\mu(t)$ & $R_X(t;\, t-\tau)$ we can find covariance of $X(t_1)$, $X(t_2)$ as

$$\begin{aligned} Cov(X(t_1), X(t_2)) &= E(X(t_1)X(t_2)) \\ &\qquad - E(X(t_1))E(X(t_2)) \\ &= R_X(t_1;\, t_1 - (t_1 - t_2)) \end{aligned}$$

$$- \mu(t_1)\,\mu(t_2)$$

Since Mean & Covariance characterize (joint) pdf of Gaussian random variables

$\mu(t)$ & $R_X(t, t-\tau)$ characterize Gaussian RP completely.

- Recall for Wide Sense Stationary RP

$$\mu(t) = \mu \quad \text{(constant)}$$

ACF: $E(X(t)\,X(t-\tau))$ depends only on τ

& denote $R_X(\tau) = E(X(t)\,X(t-\tau))$

- ~~IP~~ Note $\mathrm{Cov}(X(t_1)\,X(t_2))$

$$= \mathrm{Cov}(X(t_1+d)\,X(t_2+d))$$

for any t_1, t_2, d.

- If a Gaussian RP is WSS then it is also Stationary

$\{X(t)\}$ & $\{X(t-d)\}$ have same statistics for any delay d

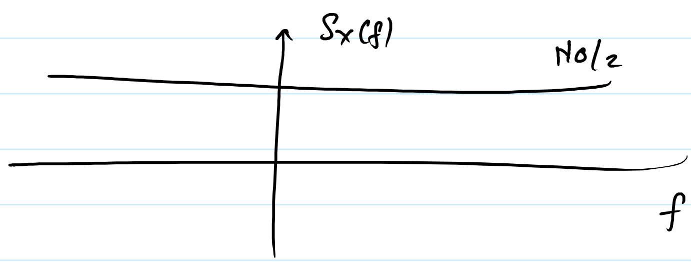
- Recall PSD for WSS

- Recall PSD for WSS

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} \, d\tau$$

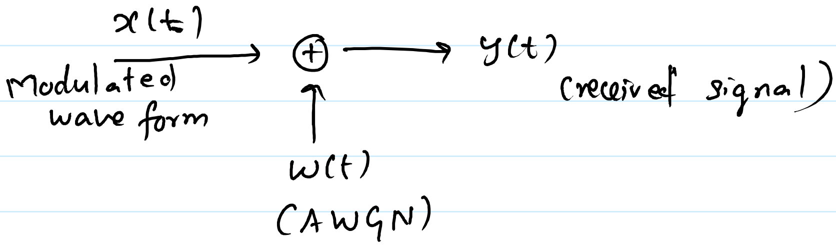
- For WSS Gaussian RP with zero mean ($\mu(t) = 0$) the PSD is equivalent characterization for RP

- A white Gaussian RP is a WSS Gaussian RP with $\mu(t) = 0$ & $S_X(f) = \text{Constant}$ for all f



- We model noise $w(t)$ in our communication system as white Gaussian Random process with PSD $N_0/2 \equiv \sigma^2$

Additive white Gaussian noise model



$w(t)$ is white Gaussian with

(two-sided) PSD $\frac{N_0}{2} \equiv \sigma^2$

- Total noise power

$$\int_{-\infty}^{\infty} S_w(f)\, df$$

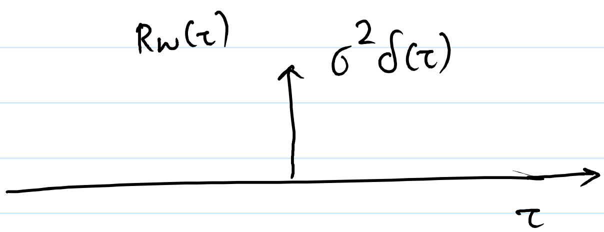
$$= \int_{-\infty}^{\infty} N_0/2 \, df$$

$$= \infty$$

- ACF of white noise

$\stackrel{\smile}{=}$ Inverse F.T. of PSD

$$R_w(\tau) = \frac{N_0}{2}\delta(\tau) = \sigma^2\delta(\tau)$$



- $E\{w(t)\, w(t-\tau)\} = 0$

if $\tau \neq 0$

$\Rightarrow E(w(t_1)\, w(t_2)) = 0$

if $t_1 \neq t_2$

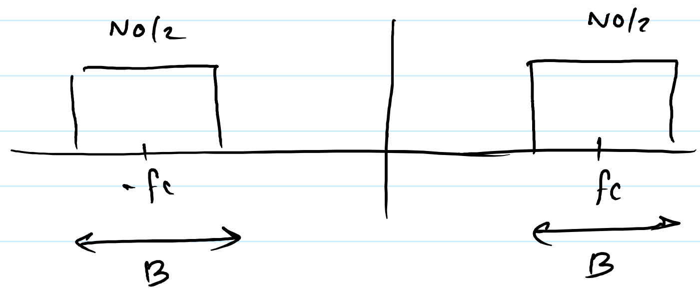
$\Rightarrow \text{Cov}\left(w(t_1), w(t_2)\right) = 0$

if $t_1 \neq t_2$

$\Rightarrow$ Samples of white noise at two different instants are independent.

- white noise with <u>infinite</u> power is a <u>simplified</u> description of noise. Noise gets filtered thru the receiver processing and the filtered noise has finite power

- After (Passband) filtering



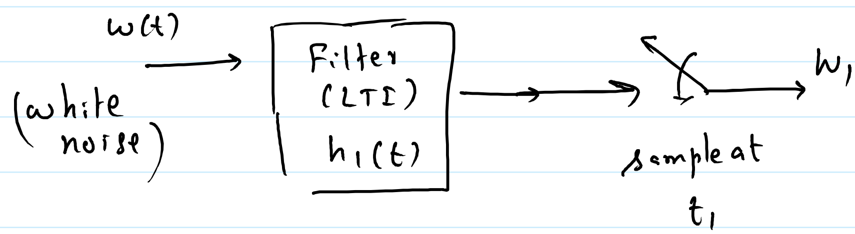
Total noise power (after filtering)

$$P_n = \frac{N_0}{2}(2B)$$

$$= N_0 . B$$

(finite power)

- Properties of filtering white noise (and sampling)



$$W_1 = \int_{-\infty}^{\infty} w(t)\, \underbrace{h_1(t_1 - t)}\, dt$$

Let $u_1(t) = h_1(t_1 - t)$

filtering + Sampling is equivalent to inner product

$$W_1 = \int_{-\infty}^{\infty} w(t)\, u_1(t)\, dt = \langle w(t), u_1(t) \rangle$$

Similarly Let $u_2(t)$ be another function

Let

$$W_2 = \int_{-\infty}^{\infty} w(t)\, u_2(t)\, dt = \langle w(t), u_2(t) \rangle$$

Claims:

- W_1 & W_2 are jointly Gaussian
- $E(W_1) = E(W_2) = 0$
- $\mathrm{Var}(W_1) = \frac{N_0}{2} \| u_1(t) \|^2$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |u_1(t)|^2\, dt$$

- $\mathrm{Var}(W_2) = \frac{N_0}{2} \| u_2(t) \|^2$

- Covariance $Cov(W_1, W_2) = \frac{N_0}{2} \langle u_1(t), u_2(t) \rangle$

- If $u_1(t)$, $u_2(t)$ are orthogonal then W_1 & W_2 are independent.

<u>Proof:</u>

$$Cov(W_1, W_2) \overset{\text{due to zero mean}}{=} E(W_1 W_2)$$

$$= E\left\{ \left(\int_{-\infty}^{\infty} w(t_1)\, u_1(t_1)\, dt_1 \right) \times \left(\int_{-\infty}^{\infty} w(t_2)\, u_2(t_2)\, dt_2 \right) \right\}$$

$$= E\left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(t_1)\, w(t_2)\, u_1(t_1)\, u_2(t_2)\, dt_1\, dt_2 \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{E\{w(t_1)\, w(t_2)\}}_{\frac{N_0}{2}\delta(t_1 - t_2)} u_1(t_1)\, u_2(t_2)\, dt_1\, dt_2$$

$$= \int_{t_1=-\infty}^{\infty} \frac{N_0}{2}\, u_1(t_1)\, u_2(t_1)\, dt_1$$

$$= \frac{N_0}{2} \langle u_1(t), u_2(t) \rangle$$