26 September 2016 08:59

Digital Demodulation (using Hypothesis Testing framework)

Consider single symbol demodulation

Symbol takes 1 out of M possibilities

received signal

$$y(t) = \underbrace{s(t)}_{\text{transmitted signal}} + \underbrace{w(t)}_{\text{AWGN}}$$

$s(t)$ takes one of M possible waveforms $\{s_1(t), s_2(t), \ldots, s_M(t)\}$

Recall, for linear modulation

Recall, for linear modulation

$$s_i(t) = b_i \, p(t)$$

$p(t)$ → pulse shape

b_i → i^{th} constellation point

b_i → takes one of M values

For non-linear modulation (orthogonal or FSK)

$$s_i(t) = \cos(2\pi f_i t)$$

f_i → frequency corresponding to i^{th} symbol

For both linear & non-linear modulation, we can build optimal demodulator using Hypothesis testing framework

Hypothesis testing for demodulation

$$
\left.
\begin{aligned}
H_1 &: \quad y(t) = s_1(t) + w(t) \\
H_2 &: \quad y(t) = s_2(t) + w(t) \\
&\quad\vdots \\
H_M &: \quad y(t) = s_M(t) + w(t)
\end{aligned}
\right\}
\text{continuous time model}
$$

$w(t) \Rightarrow$ white Gaussian noise

$s_1(t), \ldots, s_M(t)$ are deterministic & known

Given $y(t)$, find which symbol is sent.

$\times \longrightarrow$

Discrete-time Model

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}_{N \times 1} \quad \text{observation vector}$$

Under H_i : $\underline{Y} = \underline{S_i} + \underline{w}$

Signal vector $\rightarrow$ $\underline{S_i} = \begin{bmatrix} S_i(1) \\ S_i(2) \\ \vdots \\ S_i(N) \end{bmatrix}$ $\qquad$ $\underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$

$\underline{w} \rightarrow$ AWGN

$\underline{w} \rightarrow$ Coming from white Gaussian Process

$$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I}) \qquad I \rightarrow \text{identity matrix}$$

$$E\{w_i\} = 0$$

$$E\{w_i w_j\} = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases}$$

{ hypothesis testing under

- we will solve $\left\{ \begin{array}{l} \text{hypothesis testing under} \\ \text{discrete-time model} \end{array} \right.$

- Next will show ~~an~~ equivalence between continuous-time model & discrete time model

Theorem: For discrete-time AWGN model,

$$\delta_{MAP}(\underline{y}) = \underset{1 \le i \le M}{\arg\min} \; \|\underline{y} - \underline{s}_i\|^2 - 2\sigma^2 \log \pi_i$$

$\pi_i \rightarrow$ Prior probability of sending i^{th} constellation point

Proof:

Under H_i : $\underline{Y} = \underline{s}_i + \underline{w}$

Under H_i : $\underline{Y} = \underline{S_i} + \underline{w}$

$\underline{w} \sim N(\underline{0}, \sigma^2 \underline{I})$

$\underline{Y}$ is also Gaussian

Under H_i : $E(\underline{Y}) = E(\underline{S_i} + \underline{w})$

$= \underline{S_i} + E(\underline{w}) = \underline{S_i}$

Cov. of $\underline{Y} = \sigma^2 \underline{I}$

Under H_i : $\underline{Y} \sim N(\underline{S_i}, \sigma^2 \underline{I})$

Conditional pdf $f(\underline{y} | H_i) \sim N(\underline{S_i}, \sigma^2 \underline{I})$

$\delta_{MAP}(\underline{y}) = \delta_{MPE}(\underline{y})$

$= \arg\max_i \; \pi_i \, f(\underline{y} | H_i)$

$$i$$

$$= \arg\max_{i} \; \log \pi_i + \log \underbrace{f(\underline{y} \mid H_i)}_{N(\underline{s}_i,\, \sigma^2 \underline{I})}$$

$$= \arg\max_{i} \; \log \pi_i + \log \left\{ \frac{1}{(2\pi)^{N/2} \sigma^N} e^{-\frac{1}{2}(\underline{y}-\underline{s}_i)^T \frac{\underline{I}}{\sigma^2}(\underline{y}-\underline{s}_i)} \right\}$$

$$= \arg\max_{i} \; \log \pi_i + \log \frac{1}{(2\pi)^{N/2} \sigma^N} \rightarrow \text{does not depend on } i$$

$$+ \left(-\frac{1}{2\sigma^2}\right)(\underline{y}-\underline{s}_i)^T(\underline{y}-\underline{s}_i)$$

$$= \arg\max_{i} \; -\frac{1}{2\sigma^2}\left\{ \|\underline{y}-\underline{s}_i\|^2 - 2\sigma^2 \log \pi_i \right\}$$

$$= \arg\min \; \|\underline{y}-\underline{s}_i\|^2 - 2\sigma^2 \log \pi_i$$

$$\boxed{\delta_{MAP}(\underline{y}) = \arg\min_{1 \le i \le M} \|\underline{y} - \underline{s}_i\|^2 - 2\sigma^2 \log \pi_i}$$

Corollary

$$\delta_{MAP}(\underline{y}) = \arg\max_{1 \le i \le M} \langle \underline{y}, \underline{s}_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i$$

Proof:

$$\|\underline{y} - \underline{s}_i\|^2 = (\underline{y} - \underline{s}_i)^T (\underline{y} - \underline{s}_i)$$

$$= \|\underline{y}\|^2 + \|\underline{s}_i\|^2 - 2\langle \underline{y}, \underline{s}_i \rangle$$

$$\arg\min_i \|\underline{y} - \underline{s}_i\|^2 - 2\sigma^2 \log \pi_i$$

$$= \arg\min_i \underbrace{\|\underline{y}\|^2}_{} + \|s_i\|^2 - 2\langle \underline{y}, \underline{s}_i \rangle - 2\sigma^2 \log \pi_i$$

$$= \arg\min_i \; \underbrace{\cdots}_{\text{does not depend on } i} \cdots - 2\sigma^2 \log \pi_i$$

$$= \arg\min_i \; -2\left\{ \langle \underline{y}, \underline{s}_i \rangle - \frac{\|\underline{s}_i\|^2}{2} + \sigma^2 \log \pi_i \right\}$$

$$\delta_{MAP}(\underline{y}) = \arg\max_i \; \langle \underline{y}, \underline{s}_i \rangle - \frac{\|\underline{s}_i\|^2}{2} + \sigma^2 \log \pi_i$$

$$x \longrightarrow y$$

Discrete time Model

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix} \qquad \underline{S}_i = \begin{bmatrix} S_i(1) \\ S_i(2) \\ \vdots \end{bmatrix} \qquad \underline{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \end{bmatrix}$$

samples

N samples (observations)

$$\begin{bmatrix} \vdots \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad \begin{bmatrix} \vdots \\ s_i(2) \\ \vdots \\ s_i(N) \end{bmatrix} \quad \begin{bmatrix} \vdots \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

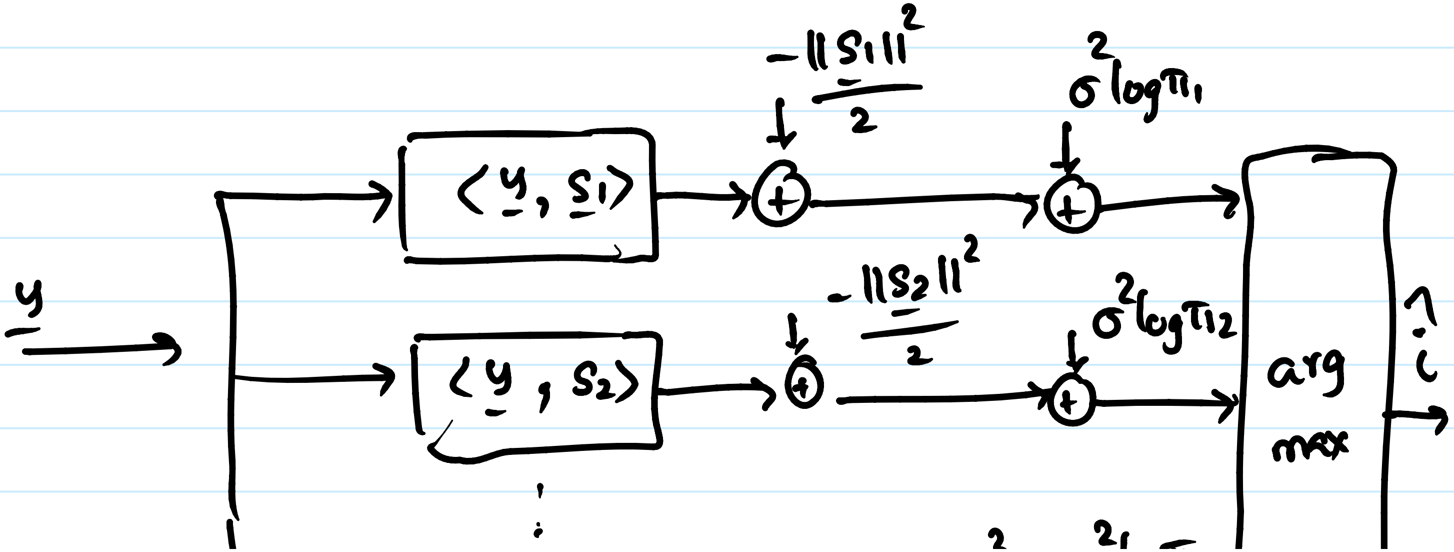
$\underline{y} = \underline{s_i} + \underline{w}$

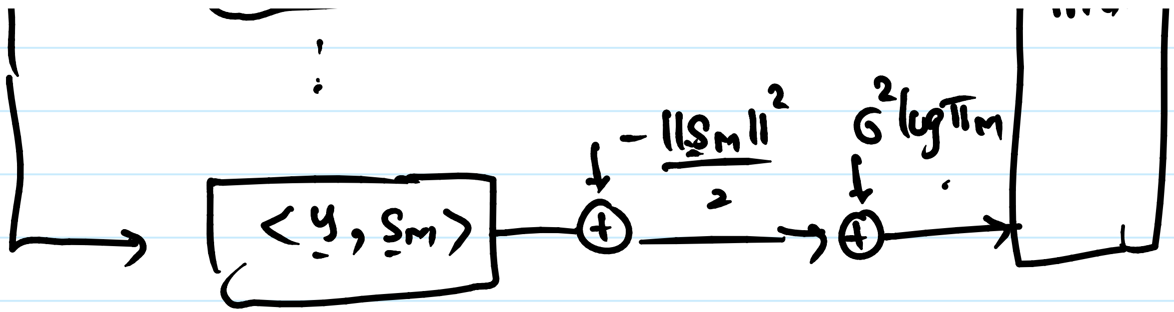
Signal vector for i^{th} symbol

WGN

M-ary Modulation : M possibilities for symbol

Given observation $\underline{y}_{N\times 1}$ optimal (min P_e) demodulator is





Guess of MAP Rule $\rightarrow \hat{i} = \delta_{MAP}(\underline{y})$

Remarks:

① Above structure is optimal for white Gaussian noise model

② $$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\langle \underline{y}, \underline{s_i} \rangle = \sum_{k=1}^{N} y_k \, s_i(k) = \underline{y}^T \underline{s_i}$$

$$\Downarrow$$

Correlation of $\underline{y}$ with $\underline{s_i}$

3 MAP Rule requires knowledge of σ^2 (Noise power)

Recall, Maximum likelihood rule

$$\delta_{ML}(\underline{y}) = \underset{1 \leq i \leq M}{\arg\max} \; f(\underline{y} \mid H_i)$$

Note if $\pi_1 = \pi_2 = \dots = \pi_M = 1/M$

then ML & MAP are same

$$\|\underline{x}\|^2 = \underline{x}^T \underline{x}$$

Theorem

$$\delta_{ML}(\underline{y}) = \underset{1 \le i \le M}{\arg\max} \ \langle \underline{y}, \underline{s}_i \rangle - \frac{\|\underline{s}_i\|^2}{2}$$

$$\delta_{ML}(\underline{y}) = \underset{1 \le i \le M}{\arg\min} \ \|\underline{y} - \underline{s}_i\|^2$$

(squared) euclidean distance between $\underline{y}$ & $\underline{s}_i$

Minimum distance decoding

Geometry of ML Rule

For the given observation $\underline{y}$, find the closest signal vector $\underline{s_i}$

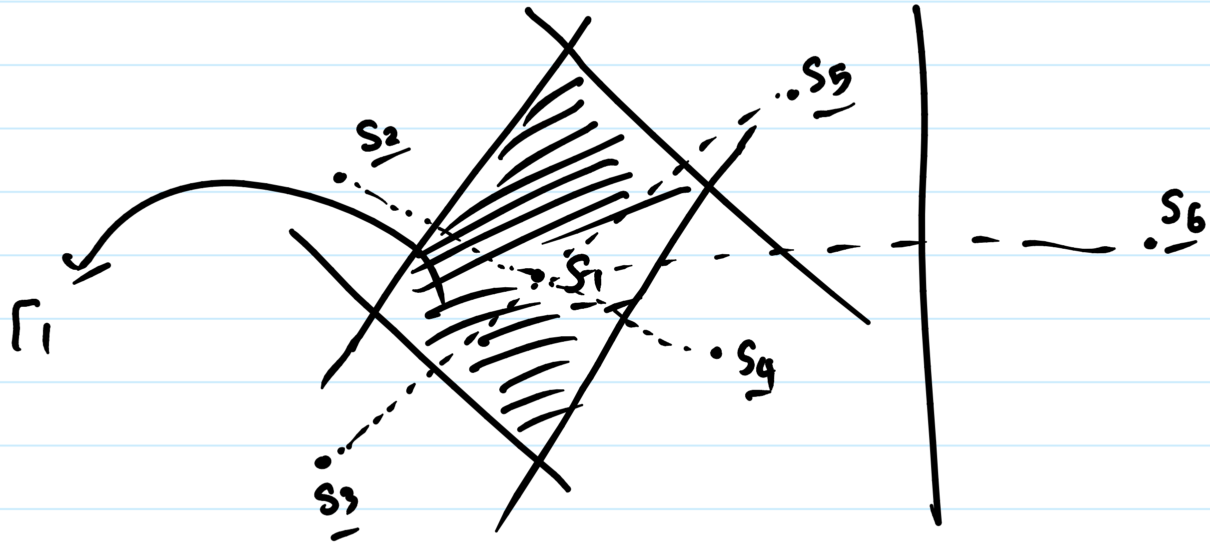
decision region Γ_i = $\left\{ \text{set of all } \underline{y} \text{ for which } \underline{s_i} \text{ is the closest signal vector} \right\}$

Example :

$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \underline{s_i} = \begin{pmatrix} s_i(1) \\ s_i(2) \end{pmatrix}$

2 Dimension

M = 6



Want to find Γ_1 (decision region in which $\underline{s_1}$ is closest to Observation)

- Consider perpendicular bisector between $\underline{s}_1$ & $\underline{s}_j$ $\quad j \neq 1$

- One of Half planes contains points closer to $\underline{s}_1$ than $\underline{s}_j$

- Intersection of all the half planes where $\underline{s}_1$ is closer than $\underline{s}_j$ $j \neq 1$

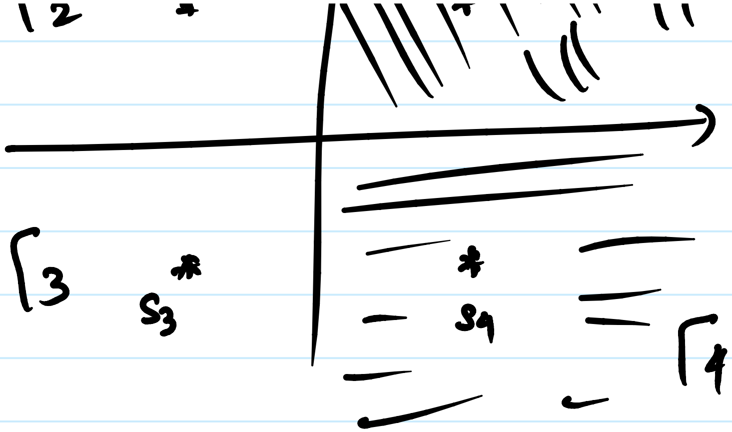
  gives decision region Γ_1

Decision Regions for regular constellations

QPSK $\qquad$ 4-Pts

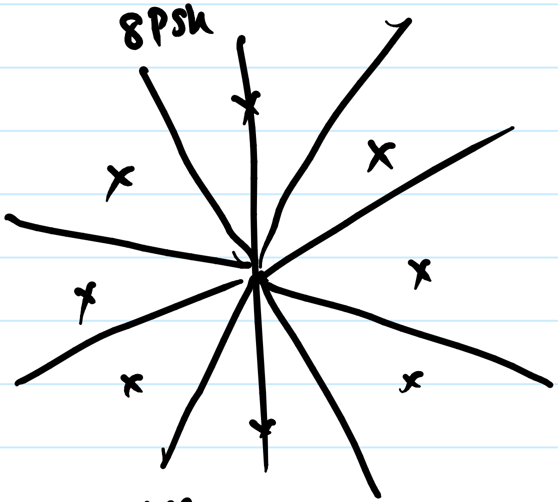
$\Gamma_1 \rightarrow$ first quadrant

quadrant



$\Gamma_4 \rightarrow$ fourth quadrant

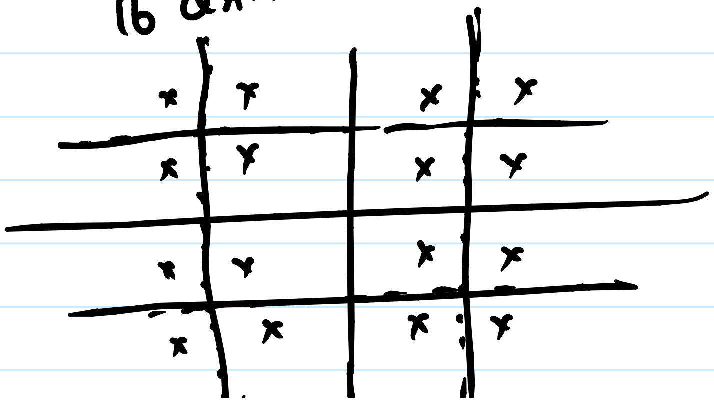
8PSK

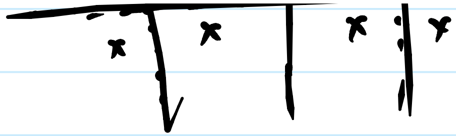


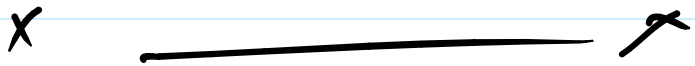
$\rightarrow$ 8 Sectors

- Each sector is corresponding decision region

16 QAM







Detection / Demodulation with continuous time observation model

Under H_i : $y(t) = s_i(t) + w(t)$

$1 \le i \le M$

$w(t) \rightarrow$ WGN

$s_i(t) \rightarrow$ Signal waveform corresponding to i^{th} symbol

i takes one out of M values

Given $y(t)$, find the optimal decision rule which minimizes average prob. of error.

Will solve this problem in 3 steps

1. Representation of continuous time signals using discrete-time samples
   (Basis Expansion in linear algebra)

already done

2. Build optimal receiver using discrete-time samples
   (Hypothesis testing framework)

3. Establish that there is <u>no loss</u> of information or optimality in going from continuous time to discrete time (if done correctly)

to discrete time (if done correctly)

(Sufficient Statistics)