Conversion from Continuous-time to Discrete time model (without any loss of information)

CT observation model:

Under H_i : $y(t) = s_i(t) + w(t)$

$i = 1, 2, \dots, M$

$w(t) \rightarrow$ white noise with PSD σ^2

Recall:

Signal set $\mathcal{S} = \{s_1(t), s_2(t), \dots, s_M(t)\}$

Signal space is the set of all signals of the form

$$a_1 s_1(t) + a_2 s_2(t) + \dots + a_M s_M(t)$$

Using Gram-Schmidt process, we find orthonormal basis for signal space

$\{\psi_1(t), \dots, \psi_N(t)\}$

Here N is dimension of signal space

Note that $N \le M$

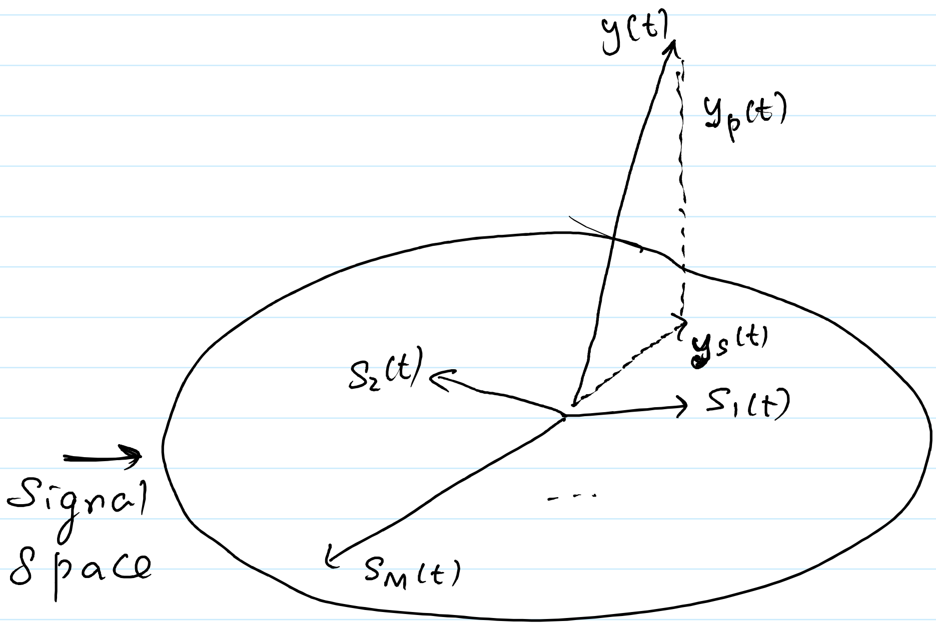
Basis Expansion:

$$s_i(t) = \sum_{n=1}^{N} a_{i,n} \psi_n(t)$$

Vector representation

$$a_{i,n} = \langle s_i(t), \psi_n(t) \rangle$$

$$\underline{s_i} = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,N} \end{bmatrix} \quad N \times 1 \text{ vector}$$



Observation $y(t) = s_i(t) + w(t)$

has a component $y_s(t)$ in

signal space and

component $y_p(t)$ which is orthogonal / perpendicular to signal space.

Define : $y_s(t) = \sum_{n=1}^{N} \langle y(t), \psi_n(t) \rangle \, \psi_n(t)$

: $y_p(t) = y(t) - y_s(t)$

Let $Y_n = \langle y(t), \psi_n(t) \rangle$

$Y_s(t) = \sum_{n=1}^{N} Y_n \, \psi_n(t)$

Under H_i :

$y(t) = s_i(t) + w(t)$

$Y_n = \langle s_i(t) + w(t), \psi_n(t) \rangle$

$= \langle s_i(t), \psi_n(t) \rangle + \underbrace{\langle w(t), \psi_n(t) \rangle}_{W_n}$

$= a_{i,n} + W_n$

Define Descrete Time Model

$$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} = \begin{bmatrix} a_{i,1} + w_1 \\ a_{i,2} + w_2 \\ \vdots \\ a_{i,N} + w_N \end{bmatrix}$$

$$\underline{r} = \underbrace{\begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,N} \end{bmatrix}}_{\underline{s}_i} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}}_{\underline{w}}$$

→ Noise in signal space

Under H_i : $\underline{r} = \underline{s}_i + \underline{w}$

Recall : Since $\{ \psi_n(t) \}$ are orthonormal

w_n's are i.i.d. Gaussian

with variance σ^2

The DT Model for hypothesis testing we solved earlier is exactly same as this.

Note: $\underline{Y}$ is vector representation

of $\underline{\underline{y_s(t)}}$

$y(t)$ has also orthogonal

component $y_p(t)$

Claim: (useful)

$y_p(t)$ has no information

for hypothesis testing

1. $y_p(t)$ is purely noise

2. $y_p(t)$ is independent of noise in signal space ($\underline{w}$)

$$\underline{w} = \begin{bmatrix} \langle w(t), \psi_1(t) \rangle \\ \vdots \\ \langle w(t), \psi_N(t) \rangle \end{bmatrix}$$

Proof:

① $$y_p(t) = y(t) - y_s(t)$$

$$= s_i(t) + w(t) - y_s(t)$$

$$= s_i(t) + w(t) - \sum_n r_n \psi_n(t)$$

$$= s_i(t) + w(t) - \sum_n (a_{i,n} + w_n)\psi_n(t)$$

$$= s_i(t) + w(t) - \underbrace{\sum_n a_{i,n}\psi_n(t)}_{s_i(t)} - \sum_n w_n \psi_n(t)$$

$$w_n = \langle w(t), \psi_n(t)\rangle$$

$$y_p(t) = w(t) - \sum_{n=1}^{N} \langle w(t), \psi_n(t)\rangle \psi_n(t)$$

$\Downarrow$

depends only on noise

& not on signal $s_i(t)$

② $y_p(t)$ is independent of

entries in $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$

$$\begin{vmatrix} \vdots \\ w_N \end{vmatrix}$$

$$W_k = \langle w(t), \psi_k(t) \rangle$$

$y_p(t)$ is Gaussian noise process

To show independence between $y_p(t)$ and W_k, we just need to show

$$E\{ y_p(t) \, W_k \} = 0 \qquad \text{(for any } t\text{)}$$

Now, $E\{ y_p(t) \, W_k \}$

$$= E\left\{ \left(w(t) - \sum_{n=1}^{N} W_n \psi_n(t) \right) W_n \right\}$$

$$= E\{ w(t) \, W_k \} - E\left\{ \sum_{n=1}^{N} W_n \psi_n(t) \, W_k \right\}$$

$$E\{ w(t) \, W_k \}$$

$$= E\{ w(t) \langle w(t), \psi_k(t) \rangle \}$$

$$= E\left\{ w(t) \int^{\infty} w(u) \, \psi_k(u) \, du \right\}$$

$$= E\left\{ w(t) \int_{-\infty}^{\infty} w(u)\, \psi_k(u)\, du \right\}$$

$$= E\left\{ \int_{-\infty}^{\infty} w(t)\, w(u)\, \psi_k(u)\, du \right\}$$

$$= \int_{-\infty}^{\infty} \underbrace{E\{w(t)\, w(u)\}}_{\sigma^2 \delta(t-u)}\, \psi_k(u)\, du$$

$$E\{w(t)\, W_k\} = \sigma^2\, \psi_k(t)$$

Now, $E\left\{ \sum_{n=1}^{N} W_n\, W_k\, \psi_n(t) \right\}$

$$= \sum_{n=1}^{N} \underbrace{E\{W_n\, W_k\}}_{\sigma^2 \delta(n-k)}\, \psi_n(t) \}$$

$$= \sigma^2\, \psi_k(t)$$

Hence $E\{ y_p(t)\, W_k \} = 0$ for any t, k

Claim: $y_p(t)$ is not irrelevant ~~useful~~ for hypothesis testing

$$y(t) = \underbrace{y_s(t)}_{\text{component in signal space}} + \underbrace{y_p(t)}_{\text{component orthogonal to signal space}}$$

Recall: hypothesis testing is done using conditional statistics

$$f(y(t) \mid H_i) = f(y_s(t), y_p(t) \mid H_i)$$

$$= f(y_s(t) \mid H_i) \cdot \underbrace{f(y_p(t) \mid H_i, y_s(t))}$$

$$= f(y_p(t))$$ due to independence

$$= f(y_s(t) \mid H_i) \quad \underbrace{f(y_p(t))}$$

is a constant term which does not depend on H_i

$y_s(t) \rightarrow$ Sufficient statistic

$y_p(t) \rightarrow$ Irrelevent statistic

$y_s(t)$ has vector representation

$$\underline{Y} = \underline{S_i} + \underline{W}$$

which is the DT model we solved already