01 November 2016 08:27

CHANNEL EQUALIZATION

- So far, we looked at demodulating a <u>single</u> symbol

- We usually transmit a <u>stream</u> of symbols

- How to (optimally) demodulate the stream of symbols?

- For that we consider linear modulation

$$s(t) = \sum_{n=-\infty}^{\infty} b(n)\, p(t-nT)$$

$1/T \rightarrow$ symbol rate

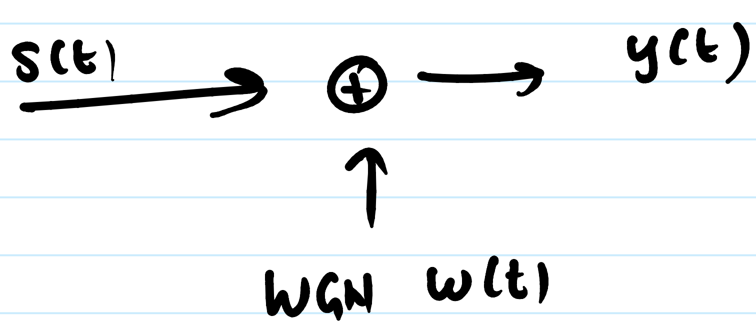
$p(t) \rightarrow$ pulse shape

$\{b(n)\} \rightarrow$ sequence (stream) of symbols

For M-ary signalling,

each $b(n)$ takes one out of M possibilities

AWGN model



$$y(t) = \sum_n b(n)\, p(t-nT) + w(t)$$

Problem: From $y(t)$, we need to

recover the transmit symbol sequence $\{b(n)\}_{n \in \mathbb{Z}}$

Assumptions

① Sequence of symbols $\{b(n), n \in \mathbb{Z}\}$ are independent

② $w(t) \rightarrow$ WGN with variance σ^2

Under ~~some~~ the following condition, demodulation of stream of symbols is straightforward (easy)

Condition: Delayed pulses $\{p(t-nT)\}_{n \in \mathbb{Z}}$

are orthonormal.

ie) $$\int_{-\infty}^{\infty} p(t-mT)\, p^*(t-kT)\, dt \rightarrow (*)$$

$$= \begin{cases} 1 & \text{if } m=k \\ 0 & \text{if } m \neq k \end{cases}$$

$m, k \in \mathbb{Z}$ (integers)

Problem

$$y(t) = S_b(t) + w(t)$$

$$S_b(t) = \sum_{n \in \mathbb{Z}} b(n)\, p(t-nT)$$

Subscript b is used to denote the dependence of signal on symbol sequence $\{b(n), n \in \mathbb{Z}\}$

Recall, with orthonormality of $\{p(t-nT),$

Recall, with orthonormality of $\{p(t-nT), n \in Z\}$ (Satisfying ★)

we can have $\{p(t-nT)\}$ as orthonormal basis for Signal Space $S_b(t)$

Let us project $y(t)$ onto each orthonormal basis function

$$\underset{\text{Sufficient Statistics}}{Z(n)} = \langle y(t), p(t-nT) \rangle \quad n \in Z$$

$$= \langle S_b(t) + w(t), p(t-nT) \rangle$$

$$= \langle S_b(t), p(t-nT) \rangle + \underbrace{\langle w(t), p(t-nT) \rangle}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad w(n)$

$$\langle s_b(t), p(t-nT)\rangle$$

$$= \left\langle \sum_{k\in\mathbb{Z}} b(k)\, p(t-kT), p(t-nT)\right\rangle$$

$$= b(n)$$ (due to orthonormality)

So

$$z(n) = b(n) + w(n), \quad n\in\mathbb{Z}$$

$z(n)$: independent sequence

$b(n)$: independent stream of data symbols

$w(n)$: independent sequence of noise samples (Projecting WGN onto orthonormal functions)

To (Optimal) demodulate $b(n)$, we ~~can~~ need to use only $z(n)$

other values $\{ z(n+1), z(n+2), \ldots$
$z(n-1), z(n-2), \ldots \}$
are not useful (irrelevant)
to demodulate $b(n)$

In other words

$b(0)$ can be demod using $z(0)$
$b(1)$ " " $z(1)$
& so on.

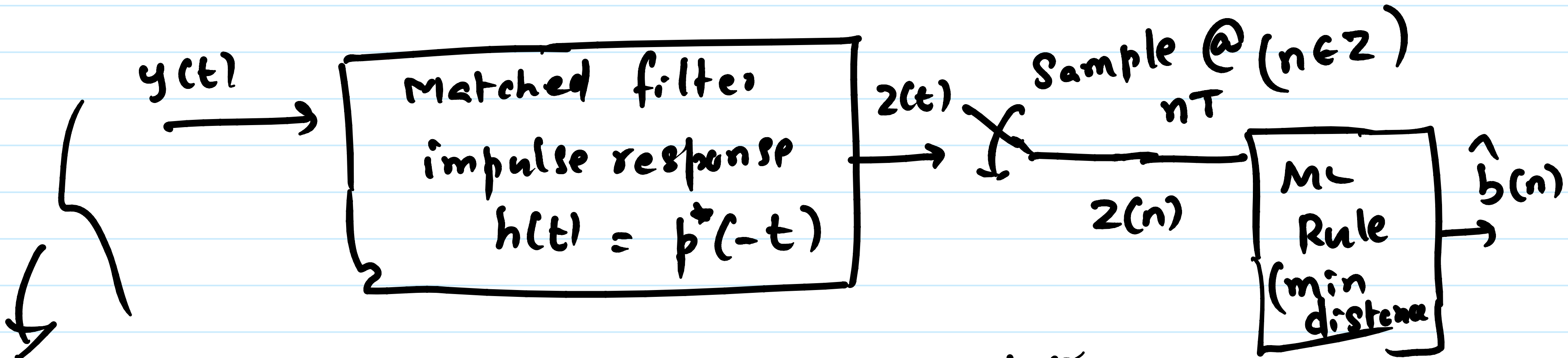
Optimal ML Rule
min. distance rule

$$\hat{b}(n) = \underset{b \in \Delta}{\arg\min} \; |z(n) - b|^2$$

$n \in \mathbb{Z}$,

$\Delta \rightarrow$ set of M-ary constellation points

$\{z(n)\}_{n \in \mathbb{Z}}$ can be obtained using Matched filter



Optimal demod for stream of symbols when delayed pulses $p(t-nT)$ are orthonormal

Convolution

$$z(t) = y(t) * h(t)$$

$$= \int_{-\infty}^{\infty} y(\tau)\, h(t-\tau)\, d\tau$$

$$z(n) = z(t)\Big|_{@\, t = nT}$$

$$= \int_{-\infty}^{\infty} y(\tau)\, h(nT-\tau)\, d\tau$$

$$= \int_{-\infty} y(\tau)\, \underbrace{h(nT-\tau)}_{p^*(\tau - nT)}\, d\tau$$

$$= \int_{-\infty}^{\infty} y(\tau)\, p^*(\tau - nT)\, d\tau$$

$$= \langle y(t), p(t-nT) \rangle$$

x ———————— x

Recall orthonormality condition

$$\int_{-\infty}^{\infty} p(t-mT)\, p^*(t-kT)\, dt$$

$m, k \in$ integers $= \begin{cases} 0 & \text{if } m \neq k \\ 1 & \text{if } m = k \end{cases}$

Let $p(t)$ satisfy the above condition

Let $P(f)$ be Fourier transform

Let $P(f)$ be Fourier transform of $p(t)$

$$p(t) \xleftrightarrow{\mathcal{F}} P(f)$$

Let $G(f) = |P(f)|^2$

If $p(t)$ satisfies orthonormality condition
(and only if)
then $G(f)$ satisfies
Nyquist criterion @
Symbol rate $1/T$

$p(t)$ is called Square-root Nyquist pulse.

× ——— ×

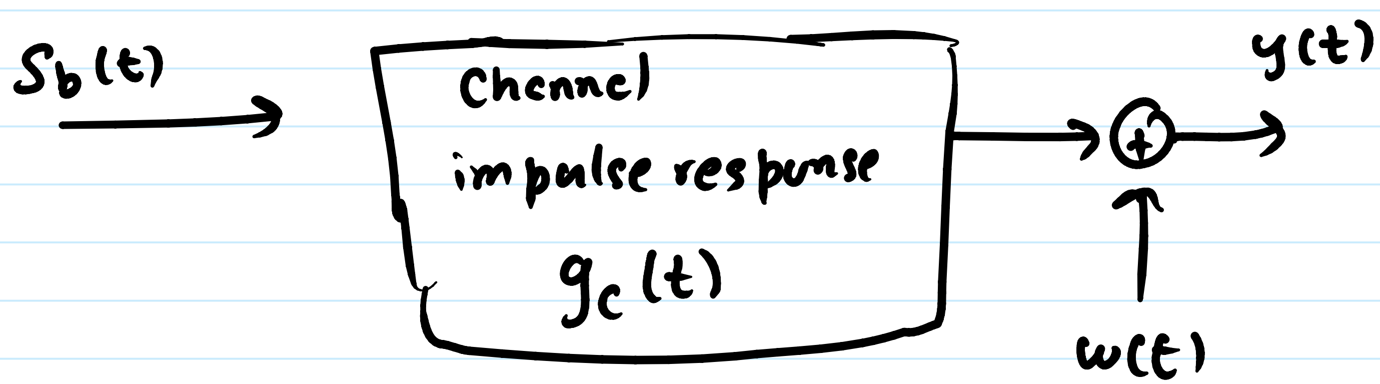
This ensures no ISI
(Inter Symbol interference)

x

In practice channel does some filtering & distorts the pulse shape

Say we use pulse $g(t)$ for lin. mod.

$$S_b(t) = \sum_{n \in \mathbb{Z}} b(n)\, g(t - nT)$$



$$y(t) = \underbrace{S_b(t) * g_c(t)}_{\text{convolution}} + w(t)$$

$$= \left(\sum_n b(n) g(t-nT) \right) * g_c(t) + w(t)$$

$$= \sum_n b(n) \left(g(t-nT) * g_c(t) \right) + w(t)$$

let $g(t) * g_c(t) = p(t)$

$$g(t-nT) * g_c(t) = p(t-nT)$$

$$y(t) = \sum_{n \in \mathbb{Z}} b(n) \underbrace{p(t-nT)}_{\text{effective pulse}} + w(t)$$

$g(t)$ $\quad$ $g_c(t)$ $\quad$ $p(t)$

$\Downarrow$ satisfies

$\Downarrow$ does not

satisfies Nyquist criterion

does not satisfy Nyquist criterion

x ——— x

Define (sampled) autocorrelation function

$$h(m) = \langle p(t), p(t-mT) \rangle$$

$m \in \mathbb{Z}$

$$= \int_{-\infty}^{\infty} p(t)\, p^*(t-mT)\, dt$$

This satisfies (from definition)

$$h(m) = h^*(-m)$$

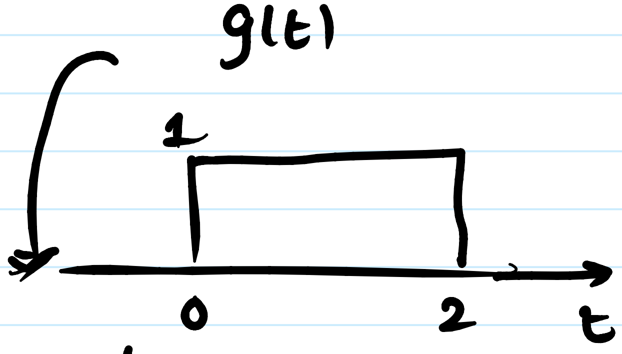
x ——— x

Running Example.

Symbol Rate $1/2 = 1/T$

Original pulse

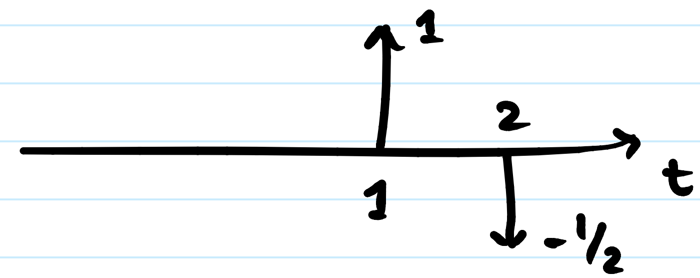
$g(t)$



delayed pulses are orthogonal (no ISI)

impulse response of channel

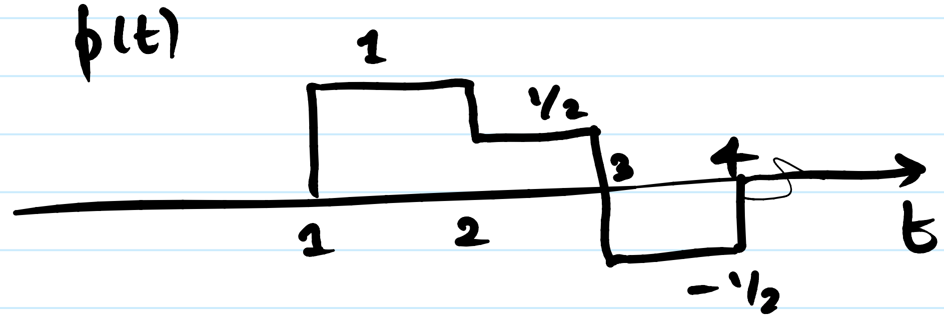
$g_c(t)$



$\delta(t-1) - \frac{1}{2}\delta(t-2)$

Effective pulse $p(t) = g(t) * g_c(t)$

$p(t)$



$T = 2$

$p(t)$ & $p(t-T)$ are not orthogonal

(Sampled) Autocorrelation function of $p(t)$

$$h(m) = \int_{-\infty}^{\infty} p(t)\, p^*(t - mT)\, dt$$

$$= \begin{cases} 3/2 & \text{if } m = 0 \\ -1/2 & \text{if } m = 1 \\ -1/2 & \text{if } m = -1 \\ 0 & \text{if } |m| \geq 2 \end{cases}$$

Since $h(m) \neq 0$ for $m = \pm 1$,

we note that

ISI exist

This means that

$b(n)$ interferes with $b(n+1)$ & $b(n-1)$

Maximum Likelihood Sequence Detection (MLSE)

Suppose $\{b(n)\}$ is the transmit sequence of symbols

Let N denote the number of symbols transmitted.

$$\underline{b} = \begin{bmatrix} \vdots \\ b(n+1) \\ b(n) \\ b(n-1) \\ \vdots \end{bmatrix}$$

Vector of transmitted symbols $(N \times 1)$

Each symbol $b(n)$ comes from M-ary constellation

Total number of possibilities

Total number of possibilities for $\underline{b}$ is M^N

$H_{\underline{b}}$: $\rightarrow$ Hypothesis corresponding to a particular transmit vector $\underline{b}$

We need to demodulate the symbols jointly when there is ISI.

There are M^N hypotheses.

Under $H_{\underline{b}}$: $y(t) = S_b(t) + \underset{\text{WGN}}{\underset{\downarrow}{w(t)}}$

$$S_b(t) = \sum_n b(n)\, p(t-nT)$$

Optimal ML rule "picks" the hypothesis for which

$S_b(t)$ is "closest" to $y(t)$.

$$\delta_{ML}(y) = \arg\min_{\underline{b}} \| y(t) - S_b(t) \|^2$$

set of all possible transmit vectors
symbol
(total number M^N)

$$\delta_{ML}(y) = \arg\max_{\underline{b}} \; \mathrm{Re}\{ \langle y(t), S_b(t) \rangle \} - \frac{\| S_b(t) \|^2}{2}$$

For $M = 4$, $N = 1000$ symbols
QPSK

total number of possible transmit

total number of possible transmit vectors is 4^{1000} (quite large)

We need to develop efficient way of implementing MLSE

$\Downarrow$ done with viterbi algorithm

Theorem:

$$\delta_{ML}(y) = \arg\max_{\underline{b}} \, Re\{\langle y(t), s_b(t)\rangle\} - \frac{\|s_b(t)\|^2}{2}$$

~~[illegible]~~ For this ML demodulation,

matched filter outputs

$$Z(n) = \langle y(t), p(t-nT)\rangle$$

$n \in \mathbb{Z}$ (integers)

$\in \mathbb{Z}$ (integers)

are sufficient statistics

Proof:

ML Rule:

$$\delta_{ML}(y) = \arg\max_{\underline{b}} \; Re\left\{\langle y(t), s_b(t)\rangle\right\} - \frac{\|s_b(t)\|^2}{2}$$

Observation $y(t)$ is involved in

computing $Re\langle y(t), s_b(t)\rangle$

$\|s_b(t)\|^2$ does not depend on $y(t)$

Now, $\langle y(t), s_b(t)\rangle$

$$= \langle y(t), \sum_n b(n)\, p(t-nT)\rangle$$

$$= \int_{-\infty}^{\infty} y(t) \left(\sum_n b(n)\, p(t-nT) \right)^* dt$$

$$= \sum_n \int_{-\infty}^{\infty} y(t)\, b^*(n)\, p^*(t-nT)\, dt$$

$$= \sum_n b^*(n) \int_{-\infty}^{\infty} y(t)\, p^*(t-nT)\, dt$$

$$= \sum_n b^*(n)\, z(n)$$

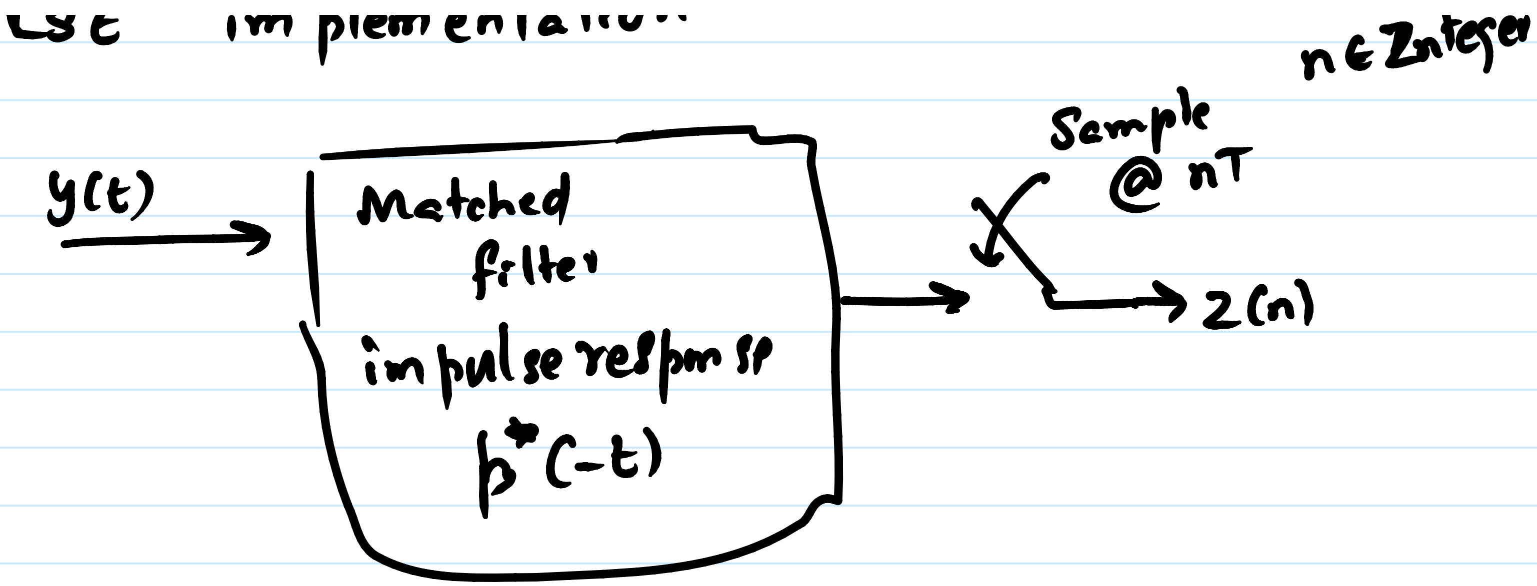
Knowing $z(n)$ is sufficient to compute ML metric.

x ———— x

MLSE implementation

$n \in \mathbb{Z}$ nteger

MLSE implementation



Using $z(n)$ compute ML metric for each possible transmit symbol vector $\underline{b}$

Efficient Computation of ML metric

There are M^N possible transmit symbol vectors

Let $\Lambda(\underline{b})$ denote the ML metric for symbol vector $\underline{b}$

metric for symbol vector $\underline{b}$

Let $\Lambda(\underline{b})$ be denoted as cost function

We want to find $\underline{b}$ which maximizes $\Lambda(\underline{b})$

Two steps are involved in MLSE

1. Computing cost function $\Lambda(\underline{b})$ for each candidate $\underline{b}$

2. Finding the arg max of cost function

Efficient way of computing cost function

$$\Lambda(\underline{b}) = Re\{\langle y(t), s_b(t)\rangle\}$$

$$- \frac{\| s_b(t) \|^2}{2}$$

Let $\quad z(n) = \langle y(t), p(t-nT) \rangle$

$$\mathrm{Re}\,\langle y(t), s_b(t) \rangle = \underbrace{\mathrm{Re} \sum_n b^*(n)\, z(n)}$$

$\Rightarrow$ additive in n

$$\| s_b(t) \|^2 = \Big\| \sum_n b(n)\, p(t-nT) \Big\|^2$$

$$= \Big\langle \sum_n b(n)\, p(t-nT), \sum_m b(m)\, p(t-mT) \Big\rangle$$

$$= \sum_n \sum_m \int_{-\infty}^{\infty} b(n)\, p(t-nT)\, b^*(m)\, p^*(t-mT)\, dt$$

$$= \quad \int^{\infty} p(t-nT)\, p^*(t-mT)\, dt$$

$$= \sum_n \sum_m b(n)\, b^*(m) \underbrace{\int_{-\infty} p(t-nT)\, p^*(t-mT)\, dt}_{h(m-n)}$$

Recall $h(k) = \int_{-\infty}^{\infty} p(t)\, p^*(t-kT)\, dt$

$$\|S_b(t)\|^2 = \sum_n \sum_m b(n)\, b^*(m)\, h(m-n)$$

We know $h(k) = h^*(-k)$

$m<n$, $m=n$, $m>n$

$$= \sum_n \sum_{m<n} b(n)\, b^*(m)\, h(m-n)$$

$$+ \sum_n |b(n)|^2\, h(0)$$

$$+ \sum_n \sum_{m>n} b(n)\, b^*(m)\, h(m-n)$$

Interchange roles of m & n in last Summation

$$= \sum_n |b(n)|^2 h(0)$$

$$+ \sum_n \sum_{m<n} b(n)\, b^*(m)\, h(m-n) + b^*(n)\, b(m)\, h(n-m)$$

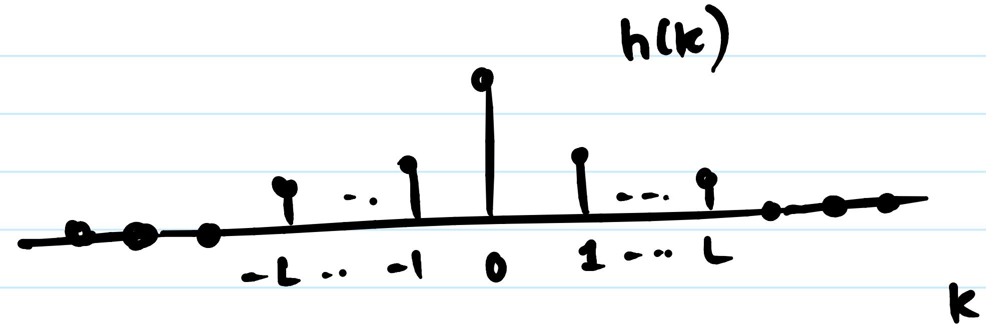
$$h(m-n) = h^*(n-m)$$

$$\|s_b(t)\|^2 = \sum_n \sum_{m<n} 2\,\mathrm{Re}\left\{ b^*(n)\, b(m)\, h(n-m) \right\}$$

$$+ \sum_n |b(n)|^2 h(0)$$

Another practical assumption

$$h(k) = 0 \quad \text{if } |k| > L$$



$$\|s_b(t)\|^2 = \sum_n |b(n)|^2 h(0) + \sum_n \sum_{m=n-1}^{n-L} 2\,\mathrm{Re}\{ b^*(n)\, b(m)\, h(n-m) \}$$

Now, cost function

$$\Lambda(\underline{b}) = \sum_n \Big\{ \mathrm{Re}\big(b^*(n)\, z(n) \big) - \frac{1}{2} h(0)\, |b(n)|^2 - \mathrm{Re}\Big(b^*(n) \sum_{m=n-1}^{n-L} b(m)\, h(n-m) \Big) \Big\}$$

additive in n

have memory

At time n, metric computation involves $b(n)$ and also

$$S(n) = [b(n-1), b(n-2), \ldots, b(n-L)]$$

↳ State at time n (M^L possible states)

$\lambda_n(b(n), S(n)) \rightarrow$ branch metric at time n

$$\lambda_n(b(n), S(n)) = \mathrm{Re}\{b^*(n)\, z(n)\} - \frac{h(0)}{2}|b(n)|^2 - \mathrm{Re}\left\{b^*(n)\sum_{k=1}^{L} b(n-k)\, h(k)\right\}$$

(upto)

Accumulated metric (upto) at time n

$$\Lambda_n(\underline{b}) = \sum_{k=0}^{n} \lambda_k(b(k), s(k))$$

$$= \Lambda_{n-1}(\underline{b}) + \lambda_n(b(n), s(n))$$

$$s(n) = [b(n-1) \ \ldots \ b(n-L)]$$

$$s(n+1) = [b(n) \ b(n-1) \ldots b(n+1-L)]$$

$\lambda_n(b(n), s(n))$ can also be thought of as state transition metric $(s(n) \rightarrow s(n+1))$

$$\lambda_n(b(n), s(n)) = \lambda_n(s(n) \rightarrow s(n+1))$$