(1) $s_0(t) = \cos\left(\frac{2\pi t}{T}\right) \qquad s_1(t) = \cos\left(\frac{4\pi t}{T}\right) \quad , 0 \le t \le T$

(a) $\psi_1(t) = \frac{s_0(t)}{\|s_0(t)\|}$

$$\|s_0(t)\|^2 = \int_0^T \cos^2\left(\frac{2\pi t}{T}\right) dt = \int_0^T \frac{1+\cos\left(\frac{4\pi t}{T}\right)}{2} dt$$

$$= \frac{T}{2} + \frac{1}{2}\left[\sin\left(\frac{4\pi t}{T}\right) \cdot \frac{T}{4\pi}\right]_0^T = \frac{T}{2}$$

$$\therefore \psi_1(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi t}{T}\right)$$

$$\phi_2(t) = s_1(t) - \langle \psi_1(t), s_1(t) \rangle \psi_1(t)$$

$$= \cos\left(\frac{4\pi t}{T}\right) - \underbrace{\left\langle \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi t}{T}\right) . \cos\left(\frac{4\pi t}{T}\right) \right\rangle}_{A} . \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi t}{T}\right)$$

$$A = \int_0^T \sqrt{\frac{2}{T}} \cdot \frac{1}{2}\left[\cos\left(\frac{6\pi t}{T}\right) + \cos\left(\frac{2\pi t}{T}\right)\right] dt$$

$$= \sqrt{\frac{2}{T}}\, \frac{1}{2}\left[\frac{T}{6\pi} \sin\left(\frac{6\pi t}{T}\right)\Big|_0^T + \frac{T}{2\pi} \sin\left(\frac{2\pi t}{T}\right)\Big|_0^T\right]$$

$$= \frac{1}{\sqrt{2T}}\, 0 = 0$$

$$\therefore \phi_2(t) = \cos\left(\frac{4\pi t}{T}\right) , \quad \|\phi_2(t)\|^2 = \frac{T}{2}$$

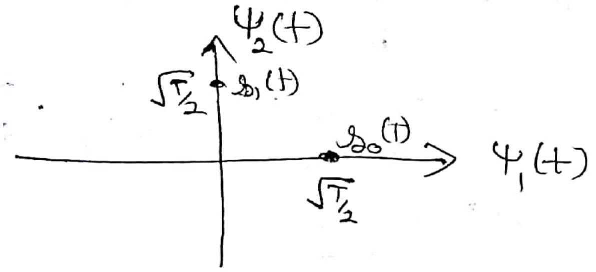
$$\psi_2(t) = \frac{\phi_2(t)}{\|\phi_2(t)\|} = \sqrt{\frac{2}{T}} \cos\left(\frac{4\pi t}{T}\right)$$

Also, $$\langle \psi_1(t), \psi_2(t) \rangle = \int_0^T \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi t}{T}\right) \cdot \sqrt{\frac{2}{T}} \cos\left(\frac{4\pi t}{T}\right) dt$$
$$= 0 \quad \text{from (A)}$$

$\boxed{\therefore \text{ set of orthonormal bases are } \psi_1(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi t}{T}\right) \text{ and } \psi_2(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{4\pi t}{T}\right)}$

(b) We can observe that,
$\boxed{s_0(t) = \sqrt{T/2}\, \psi_1(t)}$ and $\boxed{s_1(t) = \sqrt{T/2}\, \psi_2(t)}$

(c)



Signal space constellation

(d) Now, $s_0(t) = \cos\left(\frac{\pi t}{T}\right)$ $\quad s_1(t) = \cos\left(\frac{\frac{3}{2}\pi t}{T}\right)$ $\quad 0 \le t \le T$

$\psi_1(t) = \frac{s_0(t)}{\|s_0(t)\|}$, Again $\|s_0(t)\|^2 = T/2$

$\therefore \psi_1(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{\pi t}{T}\right)$

$$\phi_2(t) = s_1(t) - \langle s_1(t), \psi_1(t) \rangle\, \psi_1(t)$$

$$\langle s_1(t), \psi_1(t) \rangle = \int_0^T \cos\left(\frac{\frac{3\pi}{2} t}{T}\right) \cdot \sqrt{\frac{2}{T}} \cos\left(\frac{\pi t}{T}\right) dt$$

$$= \sqrt{\frac{2}{T}} \cdot \frac{1}{2} \int_0^T \left(\cos\left(\frac{\frac{5\pi}{2} t}{T}\right) + \cos\left(\frac{\frac{\pi}{2} t}{T}\right)\right) dt$$

$$= \frac{1}{\sqrt{2T}} \left[\sin\left(\frac{\frac{5\pi}{2} t}{T}\right) \cdot \frac{T}{\frac{5\pi}{2}} + \sin\left(\frac{\frac{\pi}{2} t}{T}\right) \cdot \frac{T}{\frac{\pi}{2}} \right]_0^T$$

$$= \frac{1}{\sqrt{2T}} \left[\frac{2T}{\pi} \cdot \frac{6}{5} \right] = \frac{\sqrt{2T}}{\pi} \cdot \frac{6}{5} \quad -(1)$$

$\sqrt{\frac{2}{T}} \int_0^T \cos\left(\frac{\frac{3\pi}{2} t}{T}\right) \cos\left(\frac{\pi t}{T}\right) = \frac{\sqrt{2T}}{\pi} \frac{6}{5}$

$\therefore \int_0^T (\cdot)\, dt = \frac{T \cdot 6}{5\pi}$

$= T \cdot \frac{k}{2}$

(*) 1

$$\therefore \phi_2(t) = \cos\left(\frac{\frac{3}{2}\pi t}{T}\right) - \underbrace{\frac{\sqrt{2T}}{\pi}\cdot\frac{6}{5}\cdot\sqrt{\frac{2}{T}}}_{\frac{12}{5\pi} = k} \cos\left(\frac{\pi t}{T}\right)$$

$$\|\phi_2(t)\|^2 = \int_0^T \left(\underbrace{\cos^2\left(\frac{\frac{3}{2}\pi t}{T}\right)}_{A} + \underbrace{k^2 \cos^2\left(\frac{\pi t}{T}\right)}_{B} - \underbrace{2k \cos\left(\frac{3\pi t}{2T}\right)\cos\left(\frac{\pi t}{T}\right)}_{C} \right) dt$$

$$A = \int_0^T \cos^2\left(\frac{\frac{3}{2}\pi t}{T}\right) dt = \int_0^T \frac{1+\cos\left(\frac{3\pi t}{T}\right)}{2}\, dt = \frac{T}{2}$$

$$B = \int_0^T k^2 \cos^2\left(\frac{\pi t}{T}\right) dt = k^2 \int_0^T \frac{1+\cos\left(\frac{2\pi t}{T}\right)}{2}\, dt = k^2 \frac{T}{2}$$

$$C = \int_0^T 2k \cos\left(\frac{3\pi t}{2T}\right)\cdot\cos\left(\frac{\pi t}{T}\right) dt$$

$$= k \int_0^T \cancel{\sin}\left(\cos\left(\frac{\frac{5\pi t}{2}}{T}\right) + \cos\left(\frac{\frac{\pi t}{2}}{T}\right) \right) dt$$

$$= k \left[\sin\left(\frac{\frac{5\pi}{2} t}{T}\right)\left(\frac{T}{\frac{5\pi}{2}}\right) + \sin\left(\frac{\frac{\pi t}{2}}{T}\right)\left(\frac{T}{\frac{\pi}{2}}\right) \right]_0^T$$

$$= k \cdot \frac{2T}{\pi} \cdot \frac{6}{5} \quad \text{— See (1)}$$

$$= \frac{12}{5\pi} \cdot k \cdot T = k^2 T$$

$$\therefore \|\phi_1(t)\|^2 = \frac{T}{2} + k^2 \frac{T}{2} - k^2 T = \frac{T}{2} - k^2\left(\frac{T}{2}\right)$$

$$= \frac{T}{2}(1-k^2)$$

$$\therefore \psi_2(t) = \frac{\cos\left(\frac{3\pi t}{2T}\right) - k\cos\left(\frac{\pi t}{T}\right)}{\sqrt{\frac{T}{2}(1-k^2)}}$$

$$\therefore \ \psi_1(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{\pi t}{T}\right)$$

$$\text{and} \ \psi_2(t) = \frac{\cos\left(\frac{3\pi t/2}{T}\right) - k\cos\left(\frac{\pi t}{T}\right)}{\left(\frac{T}{2}(1-k^2)\right)^{1/2}}$$

are the orthonormal bases

→ To find $s_0(t)$ and $s_1(t)$ in terms of $\psi_1(t)$ and $\psi_2(t)$ we find their projections on $\psi_1(t)$ and $\psi_2(t)$

i.e. $\langle s_0(t), \psi_1(t)\rangle = \int_0^T \cos\left(\frac{\pi t}{T}\right) \cdot \sqrt{\frac{2}{T}} \cos\left(\frac{\pi t}{T}\right) dt$

$$\sqrt{\frac{2}{T}} \int_0^T \cos\left(\frac{\pi t}{T}\right) \cdot \cos\left(\frac{\pi t}{T}\right) dt = \sqrt{T/2}$$

$$\int_0^T (\ ) \, dt = \frac{T}{2} \quad (*)_2$$

$$= \sqrt{\frac{2}{T}} \cdot \frac{1}{2} \int_0^T \left(1 + \cos\left(\frac{2\pi t}{T}\right)\right) dt$$

$$= \sqrt{T/2} \quad \text{— (2)}$$

$$\langle s_0(t), \psi_2(t)\rangle = \int_0^T \cos\left(\frac{\pi t}{T}\right) \cdot \frac{\cos\left(\frac{3\pi t/2}{T}\right) - k\cos\left(\frac{\pi t}{T}\right)}{\left(\frac{T}{2}(1-k^2)\right)^{1/2}} dt$$

$$= \frac{1}{\left(\frac{T}{2}(1-k^2)\right)^{1/2}} \left(\frac{T \cdot k}{2} - k \cdot \frac{T}{2}\right) = 0$$

from $(*)_s$

~~Similarly, we find~~,

$$\therefore \ s_0(t) = \langle s_0(t), \psi_1(t)\rangle \psi_1(t) + \langle s_0(t), \psi_2(t)\rangle \psi_2(t)$$

$$\boxed{s_0(t) = \sqrt{T/2} \cdot \psi_1(t) + 0 \cdot \psi_2(t)}$$

Similarly, $\langle s_1(t), \psi_1(t)\rangle = \dfrac{\sqrt{2T}}{\pi}\cdot\dfrac{6}{5}$ from ①

$$= \sqrt{2T}\cdot\frac{K}{2} = \sqrt{\frac{T}{2}}\,K$$

$$\langle s_1(t), \psi_2(t)\rangle = \int_0^T \cos\left(\frac{\frac{3\pi}{2}t}{T}\right)\cdot\frac{\cos\left(\frac{\frac{3\pi}{2}t}{T}\right) - K\cos\left(\frac{\pi t}{T}\right)}{\left(\frac{T}{2}(1-K^2)\right)^{1/2}}\,dt$$

wkT, $\displaystyle\int_0^T \cos^2\left(\frac{\frac{3\pi}{2}t}{T}\right)dt = \frac{T}{2}$ from Ⓐ

also, $\displaystyle\int_0^T \cos\left(\frac{\frac{3\pi}{2}t}{T}\right)\cdot\cos\left(\frac{\pi t}{T}\right)dt = \frac{T\cdot K}{2}$ from ⊛.1

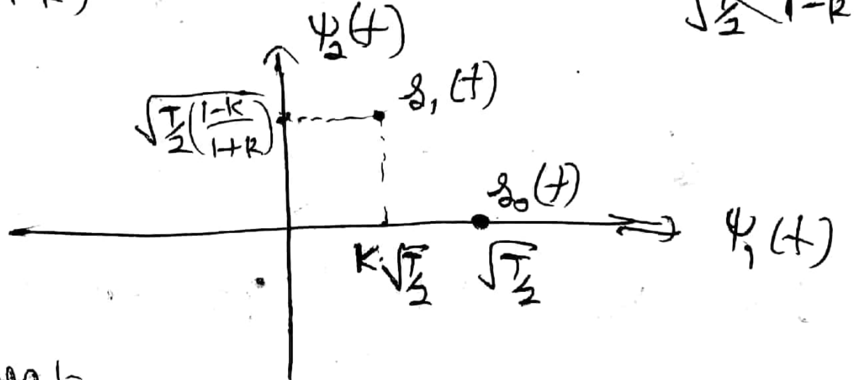
$$\therefore \langle s_1(t), \psi_2(t)\rangle = \frac{1}{\left(\frac{T}{2}(1-K^2)\right)^{1/2}}\cdot\left(\frac{T}{2} - \frac{TK}{2}\right)$$

$$= \sqrt{T/2}\,\frac{(1-K)}{(1-K^2)^{1/2}} = \sqrt{\frac{T}{2}\left(\frac{1-K}{1+K}\right)}$$

$$\boxed{\therefore s_1(t) = \sqrt{\frac{T}{2}}K\cdot\psi_1(t) + \sqrt{\frac{T}{2}\left(\frac{1-K}{1+K}\right)}\,\psi_2(t)}$$

~~Aliter~~ $\cancel{\psi_1(t) = \sqrt{\frac{2}{T}}\cos\left(\frac{\pi t}{T}\right) = \sqrt{\frac{2}{T}}\,s_0(t) \quad \therefore s_0(t) = \sqrt{\frac{T}{2}}\,\psi_1(t)}$

$\cancel{\psi_2(t) = \dfrac{\cos\left(\frac{3\pi t}{2T}\right) - K\cos\left(\frac{\pi t}{T}\right)}{\sqrt{\frac{T}{2}}(1-K^2)^{1/2}} = \dfrac{s_1(t) - K\sqrt{\frac{T}{2}}\,\psi_1(t)}{\sqrt{\frac{T}{2}}\sqrt{1-K^2}}}$



Signal space constellation

③ $r_1 = s + n_1$, s can be $s_1 = \sqrt{E_s}$ or $-\sqrt{E_s}$

n_1, n_2 iid $\sim N(0, \sigma^2)$

$$r_1 = \sqrt{E_s} + n_1 \quad \text{wp } 0.5$$
$$= -\sqrt{E_s} + n_1 \quad \text{wp } 0.5$$

$$r_2 = 0 + n_1 + n_2$$

(a). Say, $\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_1+n_2 \end{bmatrix}$

Under H_1 : $\underline{r} = \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_1+n_2 \end{bmatrix}$

Under H_0 : $\underline{r} = \begin{bmatrix} -\sqrt{E_s} \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_1+n_2 \end{bmatrix}$

Let C = Covariance matrix of $\begin{bmatrix} n_1 \\ n_1+n_2 \end{bmatrix}$

$$C = \begin{bmatrix} E(n_1^2) & E(n_1(n_1+n_2)) \\ E((n_1+n_2)n_1) & E((n_1+n_2)^2) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2+0 \\ \sigma^2+0 & \sigma^2+\sigma^2+0 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

since $E(n_1^2) = Var(n_1) = \sigma^2$ }
$E(n_2^2) = Var(n_2) = \sigma^2$ } since n_1, n_2 are zero mean

$E[n_1 n_2] = 0$ $\Rightarrow$ since n_1, n_2 are independent

$\therefore f(\underline{r} \mid H_1) \sim N\left(\begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}, C\right)$, $f(\underline{r} \mid H_0) \sim N\left(\begin{bmatrix} -\sqrt{E_s} \\ 0 \end{bmatrix}, C\right)$

$$C^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

let $m_0 = \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}$ then, $-m_0 = \begin{bmatrix} -\sqrt{E_s} \\ 0 \end{bmatrix}$

$$\cancel{\frac{1}{2\pi |C|^{1/2}}} e^{-\frac{1}{2}(\underline{r}-\underline{m}_0)^T C^{-1}(\underline{r}-\underline{m}_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \cancel{\frac{1}{2\pi |C|^{1/2}}} e^{-\frac{1}{2}(\underline{r}+\underline{m}_0)^T C^{-1}(\underline{r}+\underline{m}_0)}$$

Taking log,

$$-(\underline{r}-\underline{m}_0)^T C^{-1}(\underline{r}-\underline{m}_0) \underset{H_0}{\overset{H_1}{\gtrless}} -(\underline{r}+\underline{m}_0) C^{-1}(\underline{r}+\underline{m}_0)$$

$$\begin{pmatrix} \cancel{-\underline{r}^T C^{-1}\underline{r}} + \underline{r}^T C^{-1}\underline{m}_0 \\ + \underline{m}_0^T C^{-1}\underline{r} - \cancel{\underline{m}_0^T C^{-1}\underline{m}_0} \end{pmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} \begin{pmatrix} \cancel{-\underline{r}^T C^{-1}\underline{r}} - \underline{r}^T C^{-1}\underline{m}_0 \\ - \underline{m}_0^T C^{-1}\underline{r} - \cancel{\underline{m}_0^T C^{-1}\underline{m}_0} \end{pmatrix}$$

$$\cancel{2}(\underline{r}^T C^{-1}\underline{m}_0 + \underline{m}_0^T C^{-1}\underline{r}) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\underline{r}^T C^{-1}\underline{m}_0 = [r_1 \;\; r_2] \frac{1}{\sigma^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sigma^2}[(2r_1 - r_2) \;\; (-r_1 + r_2)] \begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}$$

$$= (2r_1 - r_2)\left(\sqrt{E_s}/\sigma^2\right).$$

$$\underline{m}_0^T C^{-1}\underline{r} = [\sqrt{E_s} \;\; 0] \frac{1}{\sigma^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$= \frac{1}{\sigma^2}[2\sqrt{E_s} \;\; -\sqrt{E_s}] \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{\sqrt{E_s}}{\sigma^2}(2r_1 - r_2)$$

$$\therefore 2\frac{\sqrt{E_s}}{\sigma^2}(2r_1 - r_2) \underset{H_0}{\overset{H_1}{\gtrless}} 0, \quad 2r_1 - r_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\downarrow$$

$$g(r_1, r_2) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

(b) The optimal rule depends on r_2, since the optimal decision rule depends on r_2

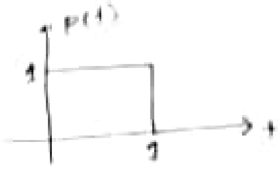
2) $p(t) = I_{[0,1]}(t)$

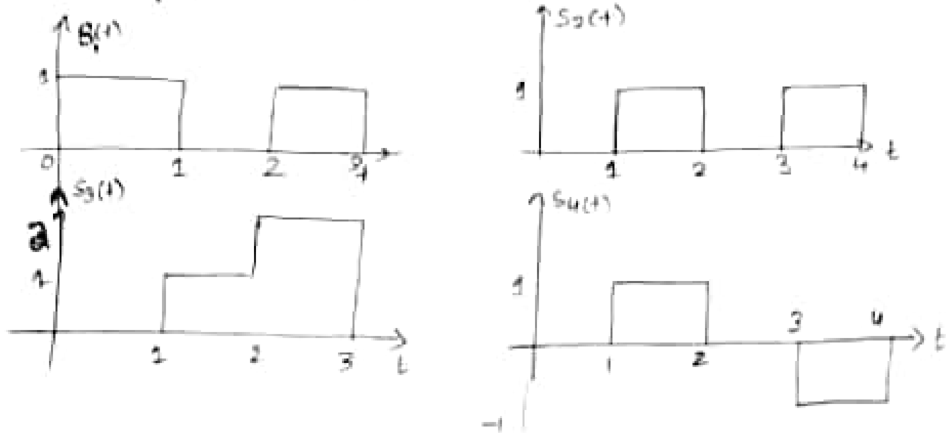
$s_1(t) = p(t) + p(t-2)$

$s_2(t) = p(t-1) + p(t-3)$

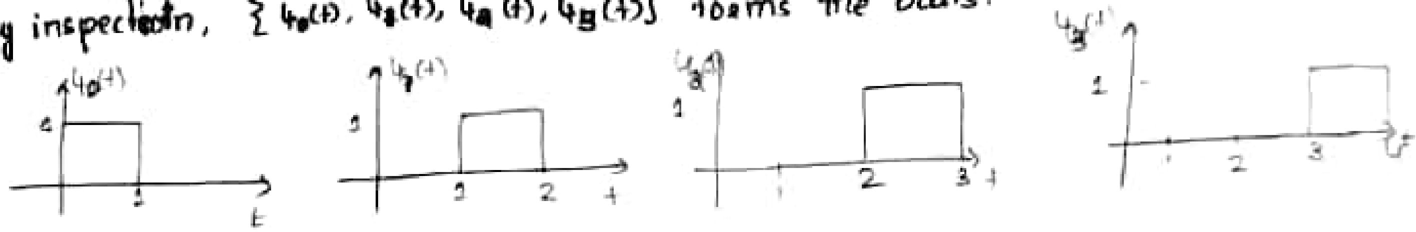
$s_3(t) = p(t-1) + 2p(t-2)$

$s_4(t) = p(t-1) - p(t-3)$





a) By inspection, $\{\psi_0(t), \psi_1(t), \psi_2(t), \psi_3(t)\}$ forms the basis.



Here, $\psi_i(t) = p(t-i)$

$\therefore$ $s_1(t) = \psi_0(t) + \psi_2(t)$

$s_2(t) = \psi_1(t) + \psi_3(t)$

$s_3(t) = \psi_1(t) + 2\psi_2(t)$

$s_4(t) = \psi_1(t) - \psi_3(t)$

$\Rightarrow$ vector representation in terms of $\{\psi_i(t)\}_{i=0 \text{ to } 3}$

$\underline{s}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$, $\underline{s}_2 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$, $\underline{s}_3 = \begin{bmatrix}0\\1\\2\\0\end{bmatrix}$, $\underline{s}_4 = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$

b) Let $\{\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)\}$ be the Basis for $\{s_i(t)\}$ using Gram-Schmidt procedure.

* $\phi_1(t) = \dfrac{s_1(t)}{\|s_1(t)\|^2}$

$\|s_1(t)\|^2 = \int_0^1 1^2 dt + \int_2^3 1^2 dt = 2$ $\Rightarrow$ $\boxed{\phi_1(t) = \frac{1}{\sqrt{2}} s_1(t)}$

* $\phi_2(t) = \dfrac{s_2(t) - \langle s_2(t), \phi_1(t)\rangle \phi_1(t)}{\| s_2(t) - \langle s_2(t), \phi_1(t)\rangle \phi_1(t) \|}$

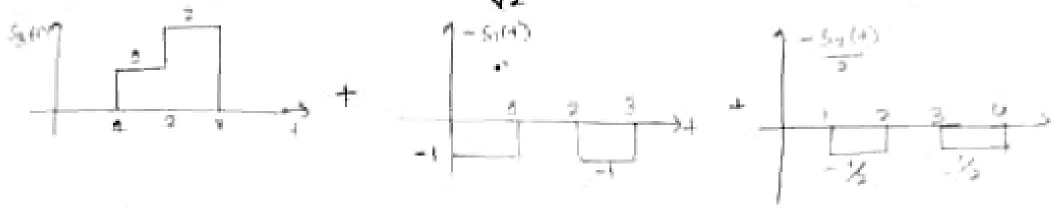
$\langle s_2(t), \phi_1(t)\rangle = 0$ & $\|s_2(t)\| = \|s_1(t)\| = \sqrt{2}$ $\Rightarrow$ $\boxed{\phi_2(t) = \frac{1}{\sqrt{2}} s_2(t)}$

* $\phi_3(t) = \dfrac{v_3(t)}{\|v_3(t)\|}$ where $v_3(t) = s_3(t) - \langle s_3(t), \phi_1(t)\rangle \phi_1(t) - \langle s_3(t), \phi_2(t)\rangle \phi_2(t)$

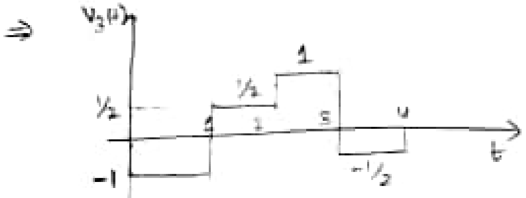
$\langle s_3(t), \phi_1(t)\rangle = \frac{1}{\sqrt{2}} \langle s_3(t), s_1(t)\rangle = \frac{1}{\sqrt{2}}\left[\int_2^3 2\,dt\right] = \sqrt{2}$

$\langle s_3(t), \phi_2(t)\rangle = \frac{1}{\sqrt{2}} \langle s_3(t), s_2(t)\rangle = \frac{1}{\sqrt{2}}$

$v_3(t) = s_3(t) - \sqrt{2}\,\phi_1(t) - \frac{1}{\sqrt{2}}\phi_2(t) = s_3(t) - s_1(t) - \frac{1}{2}s_2(t)$ —— (1)



$\Rightarrow$



—— (2)

$\Rightarrow \|v_3(t)\|^2 = (-1)^2 + (1/2)^2 + 1^2 + (-1/2)^2$

$= 2 + 1/2 = 5/2$

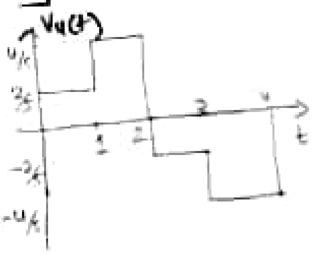
$\Rightarrow \boxed{\phi_3(t) = \sqrt{\frac{2}{5}}\, v_3(t)}$ where $v_3(t)$ is shown in fig (2). —— (3)

* $\phi_4(t) = \dfrac{v_4(t)}{\|v_4(t)\|}$ where $v_4(t) = s_4(t) - \langle s_4(t), \phi_1(t)\rangle \phi_1(t) - \langle s_4(t), \phi_2(t)\rangle \phi_2(t)$
$- \langle s_4(t), \phi_3(t)\rangle \phi_3(t)$

Here $\langle s_4(t), \phi_1(t)\rangle = 0$
$\langle s_4(t), \phi_2(t)\rangle = 0$
$\langle s_4(t), \phi_3(t)\rangle = \sqrt{\frac{2}{5}} \langle s_4(t), v_3(t)\rangle = \sqrt{\frac{2}{5}}\left[\int_1^2 \tfrac{1}{2}\,dt + \int_3^4 +\tfrac{1}{2}\,dt\right] = \sqrt{2/5}$

$\therefore$ $\cancel{\phi_4(t) = \dfrac{s_4(t)}{\|s_4(t)\|} = \dfrac{1}{\sqrt{2}} \dfrac{1}{\sqrt{2}} s_4(t)}$

$v_4(t) = s_4(t) - \sqrt{2/5}\,\phi_3(t)$
$= s_4(t) - \frac{2 \cdot v_3(t)}{5}$



$\|v_4(t)\| = \sqrt{8/5}$ —— (4)

$\boxed{\phi_4(t) = v_4(t)/\|v_4(t)\|}$

vector representation

$s_1(t) = \sqrt{2}\,\phi_1(t)$

$s_2(t) = \sqrt{2}\,\phi_2(t)$

$s_3(t) = v_3(t) + \sqrt{2}\,\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)$ (eqn (1))

$= \sqrt{\frac{5}{2}}\,\phi_3(t) + \sqrt{2}\,\phi_1(t) + \frac{1}{\sqrt{2}}\phi_2(t)$ (eqn (3))

$s_4(t) = v_4(t) + \sqrt{\frac{2}{5}}\phi_3(t)$ (eqn (4))

$= \sqrt{8/5}\,\phi_4(t) + \sqrt{2/5}\,\phi_3(t)$

$\Rightarrow$ $\underline{s}_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\underline{s}_2 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\underline{s}_3 = \begin{bmatrix} \sqrt{2} \\ 1/\sqrt{2} \\ \sqrt{5/2} \\ 0 \end{bmatrix}$, $\cancel{\underline{s}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \end{bmatrix}}$ —— (5)

$\underline{s}_4 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2/5} \\ \sqrt{8/5} \end{bmatrix}$

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c). From part (a)

$\|s_1\|^2 = 2$, $\|s_2\|^2 = 2$, $\|s_3\|^2 = 5$, $\|s_4\|^2 = 2$

Also from ④,

$\|\bar{s}_1\|^2 = 2$, $\|\bar{s}_2\|^2 = 2$, $\|\bar{s}_3\|^2 = 5$, $\|\bar{s}_4\|^2 = 2$

ie, ~~Sma~~ ie, energies in both representations $\|s_i\|^2$ & $\|\bar{s}_i\|^2$ are equal

- Relative distance for $\{s_i\}$.

$\|s_1 - s_2\|^2 = \|[1\ -1\ 1\ -1]^T\|^2 = 4$

$\|s_1 - s_3\|^2 = \|[1\ -1\ -1\ 0]^T\|^2 = 3$

$\|s_1 - s_4\|^2 = 4.$

$\|s_2 - s_3\|^2 = 5$

$\|s_2 - s_4\|^2 = 4$

$\|s_3 - s_4\| = 5$

Relative distance for $\{\bar{s}_i\}$

$\|\bar{s}_1 - \bar{s}_2\|_2^2 = 4.$

$\|\bar{s}_1 - \bar{s}_3\|^2 = 3$

$\|\bar{s}_1 - \bar{s}_4\|^2 = 4.$

$\|\bar{s}_3 - \bar{s}_2\|^2 = \|[\sqrt{2}, \frac{-1}{\sqrt{2}}, \sqrt{\frac{5}{2}}, 0]^T\| = 5$

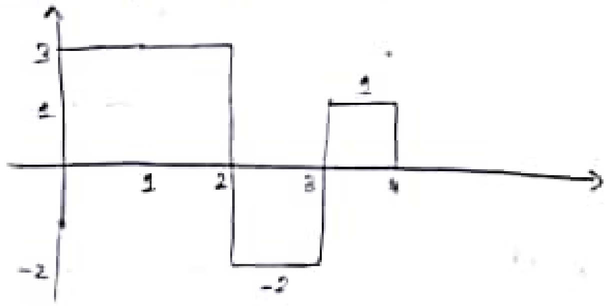
$\|\bar{s}_2 - \bar{s}_4\|^2 = 4.$

$\|s_3 - s_4\|^2 = \|[\sqrt{2}, 1/\sqrt{2}, \sqrt{5/2} - \sqrt{2/5}, \sqrt{8/5}]^T\|$

$= 2 + 1/2 + \frac{9}{10} + \frac{8}{5} = 5$

Relative distances for $\{s_i\}$ & $\{\bar{s}_i\}$ are same.

d) $s(t) = 3p(t) + 3p(t-1) - 2p(t-2) + p(t-3)$



$\langle s(t), \phi_1(t)\rangle = \frac{1}{\sqrt{2}}\langle s(t), s_1(t)\rangle$

$= \frac{1}{\sqrt{2}}\left[3\int_0^2 dt - 2\int_2^3 dt\right]$

$= 4/\sqrt{2}$

$\langle s(t), \phi_2(t)\rangle = \frac{1}{\sqrt{2}}\langle s(t), s_2(t)\rangle$

$= \frac{1}{\sqrt{2}}\left[\int_1^2 3dt + \int_3^4 1dt\right] = 2\sqrt{2}$

$\langle s(t), \phi_3(t)\rangle = \sqrt{\frac{2}{5}}\langle s_3(t), v_3(t)\rangle = \sqrt{\frac{2}{5}}\left[-3\int_0^1 dt + \frac{9}{2}\int_1^2 dt + -2\int_2^3 dt + -1/2\int_3^4 dt\right]$

$= \sqrt{\frac{2}{5}}[-4] = -4\sqrt{2/5}$

$\langle s(t); \phi_4(t)\rangle = \sqrt{8/5}\langle s(t), v_4(t)\rangle = \sqrt{\frac{8}{5}}\left[\frac{6}{5} + \frac{12}{5} + \frac{4}{5} - \frac{4}{5}\right] = \frac{18}{5}\sqrt{\frac{8}{5}}$

$\Rightarrow s(t) = \frac{1}{\sqrt{2}}\phi_1(t) + 2\sqrt{2}\phi_2(t) - 4\sqrt{\frac{2}{5}}\phi_3(t) + \frac{18}{5}\sqrt{\frac{8}{5}}\phi_4(t)$

$\therefore\ \bar{s} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 2\sqrt{2} \\ -4\sqrt{\frac{2}{5}} \\ \frac{18}{5}\sqrt{\frac{8}{5}} \end{bmatrix}$