EE 5140: Tutorial 3

Signal Space Concepts, Hypothesis Testing

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1. A binary communication system uses the following two waveforms for signaling.

$$s_0(t) = \cos(\tfrac{2\pi t}{T}),\ s_1(t) = \cos(\tfrac{4\pi t}{T})\ (0 \le t \le T)$$

   (a) Find a set of orthonormal bases for these waveforms.

   (b) Express $s_0(t)$ and $s_1(t)$ as a linear combination of the bases found in (a).

   (c) Sketch a signal space (constellation) representation of $s_0(t)$ and $s_1(t)$.

   (d) Repeat $(a) - (c)$ for $s_0(t) = \cos(\frac{\pi t}{T})$, $s_1(t) = \cos(\frac{3\pi t}{2T})$ $(0 \le t \le T)$.

2. Let $p(t) = I_{[0,1]}(t)$ denote a rectangular pulse of unit duration (ranging from $t = 0$ to $t = 1$). A digital communication system uses the following waveforms for signaling: $s_1(t) = p(t) + p(t-2)$, $s_2(t) = p(t-1) + p(t-3)$, $s_3(t) = p(t-1) + 2p(t-2)$, $s_4(t) = p(t-1) - p(t-3)$

   (a) Find a set of orthonormal bases for $\{s_i(t)\}$, $i = 1, 2, 3, 4$, just by inspection, without any computations. Write down the resulting vector representations of the signals $\{s_i(t)\}$, denote them as $\{\underline{s}_i\}$, $i = 1, 2, 3, 4$.

   (b) Using Gram-Schmidt orthogonalization procedure (starting with $s_1(t)$ and going in sequence), find a set of orthonormal bases for the signal space spanned by $\{s_i(t)\}$, $i = 1, 2, 3, 4$. Find the resulting vector representation of the signals $\{s_i(t)\}$, denote them as $\{\underline{\tilde{s}}_i\}$, $i = 1, 2, 3, 4$.

   (c) Find and compare the energies (squared distances from the origin) in both the representations $\|\underline{s}_i\|^2$ and $\|\underline{\tilde{s}}_i\|^2$ for $(i = 1, 2, 3, 4)$. Also compare the relative (squared) distances in both representations, $\|\underline{s}_k - \underline{s}_l\|^2$ and $\|\underline{\tilde{s}}_k - \underline{\tilde{s}}_l\|^2$, where $k, l \in \{1, 2, 3, 4\}$.

   (d) Find the vector representation for $s(t) = 3p(t) + 3p(t-1) - 2p(t-2) + p(t-3)$ using the bases found in part (b).

3. `Irrelevant statitstics:` A receiver in a digital communication system has two received outputs r_1 and r_2 available for decision making, where

$$r_1 = s + n_1 \text{ and } r_2 = n_1 + n_2$$

Assume that the transmitted symbol s can be one of $s_1 = +\sqrt{E_s}$ or $s_2 =$ -$\sqrt{E_s}$ with equal probability. Assume that n_1 and n_2 are iid Gaussian random variables with zero mean and variance $= \sigma^2$. Also assume that s, n_1 and n_2 are independent.

(a) Derive the optimum (MAP/ML) decision rule.

(b) Does the optimal rule depend on r_2? Give an explanation for your answer.

Note: Simplify the optimum decision rule to the form: $\hat{s} = s_1$ if $g(r_1, r_2) \geq 0$ and $\hat{s} = s_2$ if $g(r_1, r_2) < 0$, where $\hat{s}$ is the decoded symbol. You need to find the function $g(\cdot, \cdot)$