Prob 2.4.1 Given $r = ab+n$

$a \sim N(0, \sigma_a^2)$
$b \sim N(0, \sigma_b^2)$
$n \sim N(0, \sigma_n^2)$

(i) $\hat{a}_{map} = \arg\max_a P(a|r) = \arg\max_a \frac{P(r|a)P(a)}{\int P(r/a)P(a)\,da}$

$$P(r|a) = \left[2\pi(a^2\sigma_b^2+\sigma_n^2)\right]^{-1/2} \exp\left[-\frac{1}{2}\frac{r^2}{(a^2\sigma_b^2+\sigma_n^2)}\right]$$

$$P(a) = (2\pi\sigma_a^2)^{-1/2} \exp\left[-\frac{1}{2}\frac{a^2}{\sigma_a^2}\right]$$

MAP Equation written as.

$$\frac{\partial}{\partial a}\left[\ln P(r|a) + \ln P(a)\right] = 0$$

or $$\frac{\partial}{\partial a}\left[\ln\left[2\pi(a^2\sigma_b^2+\sigma_n^2)\right]^{-1/2} - \frac{1}{2}\frac{r^2}{(a^2\sigma_b^2+\sigma_n^2)} - \ln(2\pi\sigma_a^2)^{-1/2} - \frac{1}{2}\frac{a^2}{\sigma_a^2}\right] = 0$$

or $$-\frac{1}{2}\frac{\partial}{\partial a}\ln\left[2\pi(a^2\sigma_b^2+\sigma_n^2)\right] - \frac{1}{2}\frac{r^2(-1)\cdot\sigma_b^2\cdot 2a}{(a^2\sigma_b^2+\sigma_n^2)^2} - 0 - \frac{1}{2}\frac{2a}{\sigma_a^2} = 0$$

$$-\frac{1}{2}\frac{1\cdot 2\pi\,\sigma_b^2\cdot 2a}{2\pi(a^2\sigma_b^2+\sigma_n^2)} + \frac{1}{2}\frac{r^2\sigma_b^2\,2a}{(a^2\sigma_b^2+\sigma_n^2)} - \frac{2a}{2\sigma_a^2} = 0$$

$$a\left[\frac{\sigma_b^2}{(a^2\sigma_b^2+\sigma_n^2)} + \frac{r^2\sigma_b^2}{(a^2\sigma_b^2+\sigma_n^2)} - \frac{1}{\sigma_a^2}\right] = 0$$

or $$\hat{a}_{MAP} = \pm\left\{\frac{\sigma_a^2\sigma_b^2 \pm \sqrt{(\sigma_a^2\sigma_b^2)^2 - 4\sigma_a^2\sigma_b^2 r^2}}{2\sigma_b^2} - \frac{\sigma_n^2}{\sigma_b^2}\right\}^{+1/2}$$

(ii) $$P(r|a,b) = (2\pi\sigma_n^2)^{-1/2}\exp\left[-\frac{(r-ab)^2}{2\sigma_n^2}\right]$$

$P(a,b) = P(a)\,P(b)$ (a & b are independent)

$$\ln P(a,b|r) = \ln(2\pi\sigma_n^2)^{-1/2} - \frac{(r-ab)^2}{2\sigma_n^2} + \ln(2\pi\sigma_a^2)^{-1/2} - \frac{1}{2}\frac{a^2}{\sigma_a^2} + \ln(2\pi\sigma_b^2)^{-1/2} - \frac{1}{2}\frac{b^2}{\sigma_b^2}$$

$$\frac{\partial}{\partial a}\ln P(a,b|r) = \frac{(r-ab)b}{\sigma_n^2} - \frac{a}{\sigma_a^2} = 0 \quad -①$$

$$\frac{\partial}{\partial b}\ln P(a,b|r) = \frac{(r-ab)a}{\sigma_n^2} - \frac{b}{\sigma_b^2} = 0 \quad -②$$

by solving eq ① & ②

$$\hat{a}_{MAP} = \frac{\sigma_a\sigma_b r - \sigma_n^2}{\sigma_b^2} \quad \& \quad \hat{b}_{MAP} = \frac{\sigma_a\sigma_b r - \sigma_n^2}{\sigma_a^2}$$

(iii) Given $r = a + \sum_{i=1}^{k} b_i + n$

(a) $$P(r|a) = \left[2\pi\left(\sum_i \sigma_{b_i}^2 + \sigma_n^2\right)\right]^{-1/2} \exp\left[-\frac{(r-a)^2}{2\left(\sum \sigma_{b_i}^2 + \sigma_n^2\right)}\right]$$

& $$P(a) = (2\pi\sigma_a^2)^{-1/2} \exp\left(-\frac{a^2}{2\sigma_a^2}\right)$$

$$\ln P(r|a) + \ln P(a) = \ln\left(2\pi\left(\sum_i \sigma_{b_i}^2 + \sigma_n^2\right)\right)^{-1/2} - \frac{(r-a)^2}{2\left(\sum \sigma_{b_i}^2 + \sigma_n^2\right)}$$
$$+ \ln(2\pi\sigma_a^2)^{-1/2} - \frac{a^2}{2\sigma_a^2}$$

Using MAP Eq.

$$\frac{\partial}{\partial a}\left[\ln P(r|a) + \ln P(a)\right] = 0$$

$$\boxed{\hat{a}_{MAP} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_n^2 + \sum^{k} \sigma_{b_i}^2}\, r}$$

(b) $$P(r|a, b_i) = \left[2\pi\left(\sum_{\substack{j=1\\ j\neq i}}^{k} \sigma_{b_j}^2 + \sigma_n^2\right)\right]^{-1/2} \exp - \left[\frac{1}{2}\,\frac{(r-a-b_i)^2}{\sum_{\substack{j=1\\ j\neq i}}^{k} \sigma_{b_j}^2 + \sigma_n^2}\right]$$

$$P(a, b_i) = \frac{1}{2\pi\,\sigma_a\sigma_{b_i}} \exp\left[-\frac{a^2}{2\sigma_a^2} - \frac{b_i^2}{2\sigma_{b_i}^2}\right]$$

$$\frac{\partial}{\partial a}\left[\ln P(r|a, b_i) + \ln P(a, b_i)\right] = 0 \qquad -(1)$$

$$\frac{\partial}{\partial b_i}\left[\ln P(r|a, b_i) + \ln P(a, b_i)\right] = 0 \qquad -(2)$$

Gives the Solution.

$$\boxed{\hat{a}_{MAP} = \frac{\sigma_a^2\, r}{\sigma_a^2 + \sum_{i=1}^{k} \sigma_{b_i} + \sigma_n^2}}$$

Prob 2.4.3 Given $r = a + n$ where $n \sim N(0, \sigma_n^2)$ and

(i) $$P_{r|a}(R|A) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \operatorname{Exp}\left[-\frac{(R-A)^2}{2\sigma_n^2}\right]$$

and $P_a(A) \sim N\left(m_0, \frac{\sigma_n^2}{K_0}\right)$

$$P_{a|r}(A|R) = \frac{P_{r|a}(R|A)\, P_a(A)}{\int P_{r|a}(R|A)\, P_a(A)\, dA}$$

$$= \frac{\cancel{\frac{1}{\sqrt{2\pi\sigma_n^2}}} \operatorname{Exp}\left(-\frac{(R-A)^2}{2\sigma_n^2}\right) \cancel{\frac{1}{\sqrt{2\pi\sigma_n^2|K}}} \operatorname{Exp}\left[-\frac{(A-m_0)^2}{2\sigma_n^2|K_0^2}\right]}{\int \cancel{\frac{1}{\sqrt{2\pi\sigma_n^2}}} \operatorname{Exp}\left(-\frac{(R-A)^2}{2\sigma_n^2}\right) \cancel{\frac{1}{\sqrt{2\pi\sigma_n^2|K}}} \operatorname{Exp}\left[-\frac{(A-m_0)^2}{2\sigma_n^2/K_0^2}\right) dA}$$

$$= \frac{\cancel{\operatorname{Exp}\left(-\frac{R^2}{2\sigma_n^2}\right)} \operatorname{Exp}\left[-\frac{A^2-2AR}{2\sigma_n^2} - \frac{A^2-2Am_0}{2\sigma_n^2|K_0^2}\right] \cancel{\operatorname{Exp}\left[\frac{m_0^2}{2\sigma_n^2|K_0^2}\right]}}{\int \cancel{\operatorname{Exp}\left(-\frac{R^2}{2\sigma_n^2}\right]} \operatorname{Exp}\left(-\frac{A^2-2AR}{2\sigma_n^2} - \frac{A^2-2Am_0}{2\sigma_n^2/K_0^2}\right) \cancel{\operatorname{Exp}\left(-\frac{m_0^2}{2\sigma_n^2|K_0^2}\right)} dA}$$

$$= \frac{\operatorname{Exp}\left[-\frac{1}{2}\, \frac{A^2\left(\sigma_n^2 + \frac{\sigma_n^2}{K_0}\right) - 2A\left(\sigma_n^2 m_0 + \frac{\sigma_n^2}{K_0^2}R\right)}{\sigma_n^2\, \sigma_n^2|K_0^2}\right]}{\int \operatorname{Exp}\left[-\frac{1}{2}\, \frac{A^2(\sigma_n^2 + \sigma_n^2|K_0^2) - 2A\left(\sigma_n^2 m_0 + \frac{\sigma_n^2}{K_0^2}R\right)}{\sigma_n^2\,\sigma_n^2|K_0^2}\right] dA}$$

$$= \frac{\operatorname{Exp}\left[-\frac{1}{2}\left(\frac{A^2 - 2A(m_0K_0^2+R)|(1+K_0^2)}{\sigma_n^2|(1+K_0^2)}\right)\right]}{\int \operatorname{Exp}\left[-\frac{1}{2}\, \frac{A^2 - 2A(m_0K_0^2+R)/(1+K_0^2)}{\sigma_n^2/(1+K_0^2)}\right] dA}$$

assume $\sigma_1^2 = \frac{\sigma_n^2}{1+K_0^2}$ & $m_1 = \frac{m_0K_0^2 + R}{1+K_0^2}$

$$P_{a|r}(A|R) = \frac{\frac{1}{\sqrt{2\pi\sigma_1^2}} \operatorname{Exp}\left[-\frac{1}{2\sigma_1^2}\left(A^2 - 2Am_1 + m_1^2 - \cancel{m_1^2}\right)\right]}{\int \frac{1}{\sqrt{2\pi\sigma_1^2}} \operatorname{Exp}\left[-\frac{1}{2\sigma_1^2}\left(A^2 - 2Am_1 + m_1^2 - \cancel{m_1^2}\right)\right] dA}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \operatorname{Exp}\left[-\frac{1}{2\sigma_1^2}(A-m_1)^2\right]$$

$$P_{a|r}(A|R) \sim N(m_1, \sigma_1^2)$$

(ii) $$P_{r|a}(R|A) = \frac{1}{(2\pi\sigma_n^2)^{N/2}} \operatorname{Exp}\left[-\frac{1}{2\sigma_n^2}\sum_{i=1}^{N}(R_i - A)^2\right]$$

$$P_{r|a}(R|A)\, P_a(A) = (2\pi\sigma_n^2)^{-N/2} \operatorname{Exp}\left(-\frac{1}{2\sigma_n^2}\sum_{i=1}^{N} R_i^2\right) \operatorname{Exp}\left[-\frac{1}{2\sigma_n^2}\left(NA^2 - 2A\sum_{i=1}^{N} R_i\right\}\right]$$
$$(2\pi\sigma_n^2|K_0^2)^{-1/2} \operatorname{Exp}\left(-\frac{(A-m_0)^2}{2\sigma_n^2|K_0^2}\right)$$

$$= (2\pi\sigma_n^2)^{-N/2}(2\pi\sigma_n^2|K_0^2)^{-1/2} \operatorname{Exp}\left(-\frac{1}{2\sigma_n^2}\sum_{i=1}^{N} R_i^2\right)$$
$$\operatorname{Exp} -\frac{1}{2\sigma_n^2}\left(NA^2 - 2A\lambda N + K_0^2A^2 - 2Am_0K_0^2 + m_0^2K_0^2\right)$$

$$= (2\pi\sigma_n^2)^{-N/2}(2\pi\sigma_n^2/K_0)^{-1/2} \operatorname{Exp}\left(-\frac{1}{2\sigma_n^2}\sum R_i^2\right) \operatorname{Exp}\left(-\frac{m_0^2K_0^2}{2\sigma_n^2}\right)$$
$$\operatorname{Exp}\left[-\frac{1}{2\sigma_n^2}\left\{A^2(N+K_0^2) - 2A(\lambda N + m_0K_0^2)\right\}\right]$$

$$P_{a|r}(A|R) = \frac{P_{r|a}(R|A)\ P_a(A)}{\int P_{r|a}(R|A)\, P_a(A)\, dA}$$

$$= \frac{\mathcal{E}xp\left[-\frac{(N+K_0^2)}{2\sigma_n^2}\left\{A^2 - 2A\left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right) + \left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right) - \left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right)\right\}\right]}{\int \mathcal{E}xp\left[-\frac{N+K_0^2}{2\sigma_n^2}\left\{A^2 - 2A\left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right) + \left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right) - \left(\frac{lN+K_0^2 m_0}{1+K_0^2}\right)\right\}\right] dA}$$

Assume $\sigma_N^2 = \frac{2\sigma_n^2}{N+K_0^2}$ $\quad m_N = \frac{l\cdot N + K_0^2 m_0}{1+K_0^2}$

$$P(A|R) = \frac{\frac{1}{\sqrt{2\pi\sigma_N^2}}\ \mathcal{E}xp\left[-\frac{1}{2\sigma_N^2}\left(A^2 - 2Am_N + m_N^2\right)\right]\ \cancel{\mathcal{E}xp\left(+\frac{1}{2\sigma_N^2} m_N^2\right)}}{\int \frac{1}{\sqrt{2\pi\sigma_N^2}}\ \mathcal{E}xp\left[\frac{1}{2\sigma_N^2}\left(A^2 - 2Am_N + m_N^2\right)\right] dA\ \cancel{\mathcal{E}xp\left(\frac{1}{2\sigma_N^2} m_N^2\right)}}$$

$$P(A|R) = \frac{1}{\sqrt{2\pi\sigma_N^2}}\ \mathcal{E}xp\left[-\frac{1}{2\sigma_N^2}(A-m_N)^2\right]$$

i.e $N(m_N, \sigma_N^2)$ //.

Prob 2.4.7 Given $r_1\ r_2 \dots r_n$ are iid ~~and~~ with mean m and variance σ^2.

$$V = \frac{1}{n}\sum_{j=1}^{n}\left(R_j - \sum_{i=1}^{n}\frac{R_i}{n}\right)^2 \quad -(1)$$

Assume $l = \frac{1}{n}\sum_{i=1}^{n} R_i$

then Eq (1) can be written as.

$$V = \frac{1}{n}\sum_{j=1}^{n}(R_j - l)^2 = \frac{1}{n}\sum_{j=1}^{n} R_j^2 - l^2$$

$$V = \frac{1}{n}\sum_{j=1}^{n} R_j^2 - \left(\frac{1}{n}\sum_{j=1}^{n} R_j\right)^2$$

$$E[V] = E\left[\frac{1}{n}\sum_{j=1}^{n} R_j^2\right] - E\left[\left(\frac{1}{n}\sum_{j=1}^{n} R_j\right)^2\right] \quad -(1)$$

$$E[l] = m \quad \& \quad E[l^2] = \frac{\sigma^2}{n} - m^2.$$

$$E[V] = \sigma^2 + \cancel{m}^2 - \frac{\sigma^2}{n} + \cancel{m}^2 = \sigma^2 - \sigma^2/n.$$

$$E[V] = \frac{(n-1)}{n}\sigma^2 \neq \sigma^2 \quad \text{i.e Unbaised Estimator.}$$

Prob 2.4.11 given

$$p_{r|a_1,a_2}(R|A_1,A_2) = (2\pi A_2)^{-1/2} \exp\left[-\frac{(R-A_1)^2}{2A_2}\right]$$

for n independent observation.

(1) $$p_{\mathbf{r}|a_1,a_2}(\mathbf{R}|A_1,A_2) = (2\pi A_2)^{-n/2} \exp\left[-\frac{1}{2A_2}\left(\sum_{i=1}^{n}(R_i-A_1)^2\right)\right]$$

$$\ln p_{\mathbf{r}|a_1,a_2}(\mathbf{R}|A_1,A_2) = \ln(2\pi A_2)^{-n/2} - \frac{1}{2A_2}\sum_{i=1}^{n}(R_i-A_1)^2$$

$$\frac{\partial \ln p_{\mathbf{r}|a_1,a_2}(\mathbf{R}|A_1,A_2)}{\partial A_1} = 0 - \frac{1}{2A_2}\sum_{i=1}^{n}(R_i-A_1)\cdot(-2) = 0$$

$$\boxed{\hat{A}_1 = \frac{1}{n}\sum_{i=1}^{n} R_i} \quad \text{— ①}$$

Similarly for A_2

$$\frac{\partial}{\partial A_2}\ln p_{\mathbf{r}|a_1,a_2}(\mathbf{R}|A_1,A_2) = -\frac{n}{2}\frac{1}{2\pi A_2}2\pi + \frac{1}{2A_2^2}\sum_{i=1}^{n}(R_i-A_1)^2 = 0$$

$$\boxed{\hat{A}_2 = \frac{1}{n}\sum_{i=1}^{n}(R_i-\hat{A}_1)^2} \quad \text{— ②}$$

this can be written as

$$\boxed{\hat{A}_2 = \frac{1}{n}\sum_{i=1}^{n} R_i^2 - \hat{A}_1^2} \quad \text{— ③}$$

(2) $$E[\hat{A}_1] = \frac{1}{n}\sum_{i=1}^{n}E[R_i] = A \quad \text{unbaised.}$$

$$E[\hat{A}_2] = \frac{1}{n}\sum_{i=1}^{n}E[R_i^2] - E[\hat{A}_1^2]$$

$$= \sigma^2 - \sigma^2/n = \frac{n-1}{n}\sigma^2$$

i.e baised Estimator.

(3) They are coupled because the estimation of variance is depends on the estimate of mean.

(4) $$C = E\left[\begin{pmatrix}\hat{A}_1-A_1\\ \hat{A}_2-A_2\end{pmatrix}\begin{pmatrix}\hat{A}_1-A_1\\ A_2-A_2\end{pmatrix}^T\right] = \begin{bmatrix} E[(\hat{A}_1-A)^2] & E[(\hat{A}_1-A_1)(\hat{A}_2-A_2)] \\ E[(\hat{A}_1-A_2)(\hat{A}_2-A_1)] & E[(\hat{A}_2-\hat{A}_2)^2]\end{bmatrix}$$

$$E[(\hat{A}_1-A)^2] = E[\hat{A}_1^2 - 2A_1\hat{A} + \hat{A}_1^2] = A_1^2/n.$$

$$E[(\hat{A}_2-A_2)^2] = \cancel{E[\hat{A}_2^2 - 2A_2 E[\hat{A}_2] / A_2^2]} = \cancel{E}[\hat{A}_2^2] - 2\cancel{A_2}\cancel{E}[\hat{A}_2] - \cancel{A_2^2}$$

$$= var[A_2] = var\left[\frac{1}{n}\sum(R_i^2 - 2R_i\hat{A}_1 + \hat{A}_1^2)\right]$$

$$= var\left[\frac{1}{n}\sum R_i^2 - 2\frac{1}{n}\sum R_i^2\hat{A}_1 + \hat{A}_2^2\right]$$

$$=$$

$$E\left[(\hat{A}_2 - A_2)^2\right] = \frac{1}{n^2}\sum_{i=1}^{n} var[R_i^2] - ~~2~~\, var[\hat{A}_1^2]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(E[R_i^4] - E[R_i^2]^2\right) - \left(E[\hat{A}_1^4] - E[\hat{A}_1^2]^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left(A_1^4 + 6A_1^2A_2 + 3A_2^2 - (A_2+A_1^2)^2\right) - \left[A_1^4 + 6A_1^2A_2/n + 3\frac{A_2^2}{n^2} - \left(\frac{A_2}{n}+A_1^2\right)^2\right]$$

$$= \frac{1}{n}\left[A_1^4 + 6A_1^2A_2 + 3A_2^2 - A_1^4 - A_2^2 - 2A_1^2A_1\right] - \left[A_1^4 + 6A_1^2\frac{A_2}{n} + \frac{3A_2^2}{n^2} - A_1^4 - \frac{A_2^2}{n^2} - 2A_1\frac{A_2}{n}\right]$$

$$= \frac{1}{n}\left(4A_1^2A_2 + 2A_2^2\right) - 4A_1^2A_2/n - 2A_2^2/n^2$$

$$= \frac{2(n-1)}{n^2}A_2^2$$

i.e $\hat{A}_1 \sim N(A_1, A_2/n)$ and $\hat{A}_2 \sim N\left(\frac{n-1}{n}A_2, \frac{2(n-1)}{n^2}A_2^2\right)$

So A_1 and A_2 are independent

$$E\left[(\hat{A}_1 - A_1)(\hat{A}_2 - A_2)\right] = 0$$

$$C = \begin{bmatrix} A_2/n & 0 \\ 0 & \frac{2(n-1)}{n}A_2^2 \end{bmatrix} //.$$

$\boxed{\text{pb. } 2.4.16}$

Given $f_{r_1,r_2|\rho}(r_1,r_2|\rho) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\, e^{-\left(\frac{R_1^2 - 2\rho R_1R_2 + R_2^2}{2(1-\rho^2)}\right)}$.

Estimate ρ by using n independent observations of (r_1, r_2).

(1) $\hat{\rho}_{ML} = ?$ $\hat{\rho}_{ML} = \arg\max_{\rho} f(r|\rho)$.

$$f_{r|\rho}(r|\rho) = \frac{1}{\left(2\pi(1-\rho^2)^{1/2}\right)^n}\exp\left[\frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(R_{i1}^2 - 2\rho R_{i1}R_{i2} + R_{i2}^2\right)\right]$$

$$\frac{\partial}{\partial\rho}\log f(r|\rho) = \frac{\partial}{\partial\rho}\left\{-n\log\left(2\pi(1-\rho^2)^{1/2}\right) - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(R_{i1}^2 - 2\rho R_{i1}R_{i2} + R_{i2}^2\right)\right]\Bigg|_{\rho=\hat{\rho}_{ML}} = 0$$

$$\Rightarrow \frac{-n}{2\pi(1-\rho^2)^{1/2}} \cdot 2\pi \cdot \left[\frac{1}{2(1-\rho^2)^{1/2}}(-2\rho) + \left(\frac{-\rho}{(1-\rho^2)^2}\sum_{i=1}^{n}(R_{i1}^2 + R_{i2}^2)\right)\right.$$
$$\left. + \sum_{i=1}^{n} R_{i1}R_{i2}\frac{\partial}{\partial\rho}\left[\frac{\rho}{1-\rho^2}\right]\right]_{\rho=\hat{\rho}_{ML}} = 0.$$

$$\Rightarrow \frac{n\rho}{(1-\rho^2)} + \sum_{i=1}^{n}(R_{i1}^2 + R_{i2}^2)\left[\frac{-\rho}{(1-\rho^2)^2}\right] + \frac{(1+\rho^2)}{(1-\rho^2)^2}\sum_{i=1}^{n} R_{i1}R_{i2}\Bigg|_{\rho=\hat{\rho}_{ML}} = 0.$$

$$\Rightarrow n\hat{\rho}_{ML}^3 - \hat{\rho}_{ML}^2\left(\sum_{i=1}^{n} R_{i1}R_{i2}\right) + \hat{\rho}_{ML}\left[\sum_{i=1}^{n}(R_{i1}^2 + R_{i2}^2) - 1\right] - \sum_{i=1}^{n} R_{i1}R_{i2} = 0$$

(2) Lower bound on Variance of any unbiased estimate of ρ:

$\Rightarrow$ It can be found by CR-Lower bound, given by,

$$\mathrm{Var}(\hat{\rho}-\rho) \geq \left[E\left[\frac{\partial^2}{\partial\rho^2}\log f_{\underline{r}_1,\underline{r}_2|\rho}(\underline{R}_1,\underline{R}_2|\rho)\right]\right]^{-1}.$$

$$\log f(\underline{R}_1,\underline{R}_2|\rho) = -N\log 2\pi - N\log(1-\rho^2)^{1/2} - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}(R_{1i}^2 - 2\rho R_{1i}R_{2i} + R_{2i}^2)$$

$$\frac{\partial}{\partial\rho}\log f(\underline{R}_1,\underline{R}_2|\rho) = \frac{N\rho}{1-\rho^2} - \sum_{i=1}^{n}\frac{\rho(R_{1i}^2 + R_{2i}^2) - \rho^2 R_{1i}R_{2i} - R_{1i} - R_{2i}}{(1-\rho^2)^2}.$$

$$\frac{\partial^2}{\partial\rho^2}\log f(\underline{R}_1,\underline{R}_2|\rho) = \frac{N(1-\rho^2) - \rho(-2\rho)}{(1-\rho^2)^2} - \sum_{i=1}^{n}\frac{(R_{1i}^2 + R_{2i}^2 - 2\rho R_{1i}R_{2i})(1-\rho^2)^2 - 2(1-\rho^2)(-2\rho)\left[\rho(R_{1i}^2+R_{2i}^2) - \rho^2 R_{1i}R_{2i} - R_{1i}R_{2i}\right]}{(1-\rho^2)^4}.$$

$$E\left[\frac{\partial^2}{\partial\rho^2}\log f(\underline{R}_1,\underline{R}_2|\rho)\right] = \frac{N(1+\rho^2)}{(1-\rho^2)^2} - \sum_{i=1}^{n} E\left[\frac{(1-\rho^2)^2\left[R_{1i}^2 + R_{2i}^2 - 2\rho R_{1i}R_{2i}\right] + 4\rho(1-\rho^2)\left[\rho(R_{1i}^2+R_{2i}^2) - (\rho^2+1)(R_{1i}R_{2i})\right]}{(1-\rho^2)^4}\right]$$

As the covariance matrix for a single obs. is,

$$E\left[\begin{pmatrix} R_{1i} \\ R_{2i}\end{pmatrix}\begin{pmatrix} R_{1i} & R_{2i}\end{pmatrix}\right] = E\begin{bmatrix} R_{1i}^2 & R_{1i}R_{2i} \\ R_{1i}R_{2i} & R_{2i}^2\end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1\end{bmatrix}$$ [From the given joint pdf].

$$\Rightarrow E\left[\frac{\partial^2}{\partial\rho^2}\log f(\underline{R}_1,\underline{R}_2|\rho)\right] = \frac{N(1+\rho^2)}{(1-\rho^2)^2} - \sum_{i=1}^{n}\frac{2(1-\rho^2)^3 + 4\rho(1-\rho^2)(\rho-\rho^3)}{(1-\rho^2)^4}.$$

$$= \frac{N(1+\rho^2)}{(1-\rho^2)^2} - \frac{2N(1+\rho^2)}{(1-\rho^2)^2} = -N\left[\frac{1+\rho^2}{(1-\rho^2)^2}\right].$$

$$\Rightarrow \text{var}(\hat{\rho}-\rho) \geq \left[\cancel{+}+N\frac{(1+\rho^2)}{(1-\rho^2)^2}\right]^{-1} = \frac{(1-\rho^2)}{N(1+\rho^2)}.$$

Lower Bound on the variance of unbiased estimate of ρ $\boxed{\text{var}(\hat{\rho}-\rho) \geq \frac{(1-\rho^2)^2}{N(1+\rho^2)}}$

Pb. 2.4.17

Given $E[\hat{a}(r)] = A + B(A)$.

$\hat{a}(R)$ is biased estimate.

$$\Rightarrow \int \hat{a}(R) f_{r/a}(R/A)\, dR = A + B(A).$$

$$\Rightarrow \frac{\partial}{\partial A}\left\{\int \hat{a}(R) f_{r/a}(R/A)\, dR\right\} = A + B(A).$$

$$\Rightarrow \int \frac{\partial}{\partial A} \log f_{r/a}(R/A) \cdot f_{r/a}(R/A)\, \hat{a}(R)\, dR = 1 + \frac{\partial B(A)}{\partial A}$$

$$\Rightarrow \int \left[\frac{\partial}{\partial A} \log f_{r/a}(R/A) \sqrt{f_{r/a}(R/A)}\right]\left[\hat{a}(R) \sqrt{f_{r/a}(R/A)}\right] dR = 1 + \frac{\partial B(A)}{\partial A}$$

$$\left|\int p(x) g(x)\, dx\right|^2 \leq \int p^2(x)\, dx \int q^2(x)\, dx$$

$$\Rightarrow \int\left[\left(\frac{\partial}{\partial A}\log f_{r/a}(R/A)\right)^2 \cdot f_{r/a}(R/A)\right] dR \cdot \int \hat{a}^2(R) f_{r/a}(R/A)\, dR = \left[1+\frac{\partial B(A)}{\partial A}\right]^2$$

$$\Rightarrow \text{var}(\hat{a}(R)) \geq \frac{\left[1+\frac{\partial B(A)}{\partial A}\right]^2}{E\left[\left(\frac{\partial}{\partial A}\ln f_{r/a}(R/A)\right)^2\right]}.$$

Pb. 2.4.20

Given $a = Lb$, L is non-singular matrix and a, b are vector Random Variables.

$\hat{a}_{MAP} = ?$ $\quad \hat{a}_{MS} = ?$ (in terms of $\hat{b}_{MAP}$, $\hat{b}_{MS}$).

(i) $\hat{b}_{MAP} = \arg\max_{b} f_{b/r}(B/R) = \arg\max_{L\cdot b} f_{Lb/r}(LB/R)$

As L is fullRank (non-Singular), every vector 'b' maps to a single vector a. So, the value of b ($\hat{b}_{MAP}$) that maximizes $f_{b/r}(B/R)$, its corresponding a (maximizer $f_{a/r}(A/R)$).

$$\Rightarrow \hat{a}_{MAP} = L \cdot \arg\max_{b} f_{b/r}(B/R).$$

$$\Rightarrow \boxed{\hat{a}_{MAP} = L \cdot \hat{b}_{MAP}}$$

(ii) $\hat{b}_{MMSE} = E[b/r]$

$\hat{a}_{MMSE} = E[a/r] = E[Lb/r] = L \cdot E[b/r]$

$$\boxed{\hat{a}_{MMSE} = L \cdot \hat{b}_{MMSE}}$$