21-03-11

Ch2, prob 19:-

$$H_0 : Y_K \sim N(\mu_0, \sigma_0^2), \quad K = 1, 2, \ldots, n$$

$$H_1 : Y_K \sim N(\mu_1, \sigma_1^2), \quad K = 1, 2, \ldots, n,$$

where μ_0, σ_0^2, μ_1 and σ_1^2 are known constants.

$Y_1, Y_2, \ldots, Y_n$ are iid.

(a)
$$P_{\underline{Y}/H_0}(\underline{y}/H_0) = \prod_{K=1}^{n} P_{Y_K/H_0}(y_K/H_0)$$

$$= \prod_{K=1}^{n} \frac{1}{(2\pi\sigma_0^2)^{1/2}} \cdot \exp\left\{ -\frac{(y_K-\mu_0)^2}{2\sigma_0^2} \right\}$$

$$= \frac{1}{(2\pi\sigma_0^2)^{n/2}} \cdot \exp\left\{ -\frac{\sum_{K=1}^{n}(y_K-\mu_0)^2}{2\sigma_0^2} \right\}$$

$-2 y_k \mu_1 \sigma_1$

$$\text{||ly} \quad P_{\underline{Y}/H_1}(\underline{y}/H_1) = \frac{1}{(2\pi\sigma_1^2)^{n/2}} \cdot \exp\left\{ -\frac{\sum_{K=1}^{n}(y_K-\mu_1)^2}{2\sigma_1^2} \right\}$$

$$\text{Likelihood Ratio, } L(\underline{y}) = \frac{P_{\underline{Y}/H_1}(\underline{y}/H_1)}{P_{\underline{Y}/H_0}(\underline{y}/H_0)} = \prod_{K=1}^{n} \exp\left\{ \frac{(y_K-\mu_0)^2}{2\sigma_0^2} - \frac{(y_K-\mu_1)^2}{2\sigma_1^2} \right\}$$

$$\text{i.e. } L(\underline{y}) = \exp\left\{ \sum_{K=1}^{n} \frac{y_K^2(\sigma_1^2-\sigma_0^2) + 2y_K(\mu_1\sigma_0^2-\mu_0\sigma_1^2) + (\mu_0^2\sigma_1^2-\mu_1^2\sigma_0^2)}{2\sigma_0^2\sigma_1^2} \right\}$$

$$L(\underline{y}) = \exp\left\{ \frac{\sigma_1^2-\sigma_0^2}{2\sigma_1^2\sigma_0^2} \sum_{K=1}^{n} y_K^2 \right\} \cdot \exp\left\{ \frac{\mu_1\sigma_0^2-\mu_0\sigma_1^2}{\sigma_0^2\sigma_1^2} \sum_{K=1}^{n} y_K \right\} \cdot \exp\left\{ \frac{n(\mu_0^2\sigma_1^2-\mu_1^2\sigma_0^2)}{2\sigma_0^2\sigma_1^2} \right\}$$

(b) case 1: $\mu_0 = \mu_1$ & $\sigma_1^2 > \sigma_0^2$.

$$L(\underline{y}) > \gamma'$$

$$\Leftrightarrow \quad (\sigma_1^2 - \sigma_0^2)\sum_{k=1}^{n} y_k^2 - 2\mu(\sigma_1^2 - \sigma_0^2)\sum_{k=1}^{n} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma''$$

$$\Leftrightarrow \quad \sum_{k=1}^{n}(y_k^2 - 2\mu y_k) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'''.$$

$$\Leftrightarrow \quad \sum_{k=1}^{n}(y_k - \mu)^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma.$$

<u>case 2</u>:- $\sigma_1^2 = \sigma_0^2 = \sigma^2$ and $\mu_1 > \mu_0$

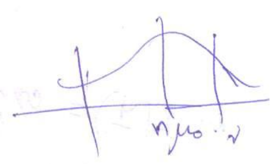
$$L(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$\Leftrightarrow \quad 2\sigma^2(\mu_1 - \mu_0)\sum y_k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma''$$

$$\Leftrightarrow \quad \boxed{\sum_{k=1}^{n} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma}$$

$$\sum_{k=1}^{n} y_k \overset{H_0}{\sim} N(n\mu_0, n\sigma_0^2)$$

$$\overset{H_1}{\sim} N(n\mu_1, n\sigma_1^2)$$



$$P_{FA} = P(\sum y_k > \gamma / H_0) = Q\left(\frac{\gamma - n\mu_0}{\sqrt{n\sigma_0^2}}\right) = \alpha$$

$$\Rightarrow \gamma = \sqrt{n\sigma_0^2}\, Q^{-1}(\alpha) + n\mu_0 \quad -(1)$$

$$P_D = P\left(\sum y_k > \gamma / H_1\right)$$

$$= Q\left(\frac{\gamma - n\mu_1}{\sqrt{n\sigma_1^2}}\right)$$

$$P_D = Q\left(\sqrt{\frac{\sigma_0^2}{\sigma_1^2}}\, Q^{-1}(\alpha) + \frac{n(\mu_0-\mu_1)}{\sqrt{n\sigma_1^2}}\right)$$

i.e. $P_D = Q\left(Q^{-1}(\alpha) - \sqrt{\frac{n}{\sigma^2}}\,(\mu_1-\mu_0)\right)$

$(y-\mu) < \sqrt{\gamma}$.

(c) $\mu_0 = \mu_1$ & $\sigma_1^2 > \sigma_0^2$, $n=1$.

Test statistic: $(y-\mu)^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$

$$\Leftrightarrow y \underset{H_0}{\overset{H_1}{\gtrless}} \mu + \sqrt{\gamma} \quad \text{or} \quad y \underset{H_0}{\overset{H_1}{\lessgtr}} \mu - \sqrt{\gamma}.$$

$$P_{FA} = P\left((y-\mu)^2 > \gamma / H_0\right)$$

$$= P\left(y > \mu + \sqrt{\gamma} / H_0\right) + P\left(y < \mu - \sqrt{\gamma} / H_0\right)$$

$$= Q\left(\frac{\sqrt{\gamma}}{\sigma_0}\right) + Q\left(+\frac{\sqrt{\gamma}}{\sigma_0}\right) = \alpha$$

$$\Rightarrow \sqrt{\gamma} = \sigma_0\, Q^{-1}(\alpha/2)$$

$$P_D = P\left((y-\mu)^2 > \gamma / H_1\right)$$

$$= P\left(y > \mu + \sqrt{\gamma} / H_1\right) + P\left(y < \mu - \sqrt{\gamma} / H_1\right)$$

$$= 2Q\left(\frac{\sqrt{\gamma}}{\sigma_1}\right) = 2Q\left(\frac{\sigma_0}{\sigma_1} Q^{-1}\left(\frac{\alpha}{2}\right)\right)$$

$$\therefore P_D = 2Q\left(\frac{\sigma_0}{\sigma_1} Q^{-1}\left(\frac{\alpha}{2}\right)\right)$$

Ch2, prob20:-

$$\mu = \mu_1 > \mu_0 = 0;$$

$$\sigma^2 = \sigma_0^2 > \sigma_1^2 .$$

$$(\sigma_1^2 - \sigma_0^2)\sum_{K=1}^{n} Y_K^2 + 2(\mu_1\sigma_0^2 - \mu_0\sigma_1^2)\sum_{K=1}^{n} Y_K \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$\text{i.e. } (\sigma_1^2 - \sigma_0^2)\sum_{K=1}^{n} Y_K^2 + 2\mu\sigma_0^2\sum_{K=1}^{n} Y_K \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

σ_0^2, μ is known, σ^2 is unknown

$$\Rightarrow \left[\left(\frac{\sigma_1^2}{2\mu\sigma_0^2}\right) - \frac{1}{2\mu}\right]\sum_{K=1}^{n} Y_K^2 + \sum_{K=1}^{n} Y_K \underset{H_0}{\overset{H_1}{\gtrless}} \gamma.$$

Since test statistic depends on unknown parameter, UMP doesn't exist.

GLRT:-

$$\hat{\sigma}_1^2 = \arg\max_{\sigma_1^2} P_{\underline{y}/H_1}(\underline{y}/H_1)$$

$$\ln P_{\underline{y}/H_1}(\underline{y}/H_1) = \ln \frac{1}{(2\pi\sigma_1^2)^{n/2}} \exp\left\{-\frac{\sum (y_K - \mu_1)^2}{2\sigma_1^2}\right\}$$

$$\ln P(\underline{y}/H_1) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma_1^2) - \frac{1}{2\sigma_1^2}\sum (y_K - \mu_1)^2.$$

$$\frac{\partial \ln P(\underline{y}/H_1)}{\partial \sigma_1^2} = 0$$

$$\Rightarrow \quad -\frac{n}{2}\cdot\frac{1}{\sigma_1^2} + \frac{1}{2(\sigma_1^2)^2} \sum (y_k-\mu)^2 = 0$$

$$\Leftrightarrow \quad \Rightarrow \boxed{\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{k=1}^{n}(y_k-\mu)^2}$$

$$GLRT(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$\Rightarrow \boxed{\left(\hat{\sigma}^2_{ML} - \sigma_0^2\right)\sum_{k=1}^{n} y_k^2 + 2\mu\sigma_0^2 \sum_{k=1}^{n} y_k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma}$$

$\frac{1}{2}\cdot\frac{e^{u}}{e^{v}}$ e^{u-v}

Ch3, prob7:-

$$H_0: Y_k = N_k, \quad k=1,2,\dots,n$$

versus $H_1: Y_k = N_k + \Theta S_k, \quad k=1,2,\dots,n,$

when $\underline{N} \sim N(\underline{0}, \Sigma)$, $\underline{S}$ is known,

ⓐ $P(\Theta=+1) = P(\Theta=-1) = 0.5$

$$P_{\underline{Y}/H_0}(\underline{y}/H_0) = \frac{1}{(2\pi)^{n/2}\cdot|\Sigma|^{1/2}}\cdot \exp\left\{-\frac{\underline{y}^T\Sigma^{-1}\underline{y}}{2}\right\}$$

$$P_{\underline{Y}/H_1}(\underline{y}/H_1) = P_{\underline{Y}/H_1}(\underline{y}/H_1, \Theta=+1)\cdot 0.5 + P_{\underline{Y}/H_1}(\underline{y}/H_1,\Theta=-1)\cdot 0.5$$

$$= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\cdot\frac{1}{2}\left[\exp\left\{-\frac{(\underline{y}-\underline{s})^T\Sigma^{-1}(\underline{y}-\underline{s})}{2}\right\} + \exp\left\{-\frac{(\underline{y}+\underline{s})^T\Sigma^{-1}(\underline{y}+\underline{s})}{2}\right\}\right]$$

$$\Rightarrow L(\underline{y}) = \cancel{\frac{1}{2}\exp\left\{\frac{(\underline{y}^T\Sigma^{-1}\underline{y}) - [\underline{y}^T\Sigma^{-1}\underline{y} - \underline{y}^T\Sigma^{-1}\underline{s} - \underline{s}^T\Sigma^{-1}\underline{y} + \underline{s}^T\Sigma^{-1}\underline{s}}{2} + \underline{y}^T\Sigma^{-1}\underline{y} + \underline{y}^T\Sigma^{-1}\underline{s} + \underline{s}^T\Sigma^{-1}\underline{y} + \underline{s}^T\Sigma^{-1}\underline{s}]\right\}}$$

$$\cancel{= \frac{1}{2}\exp\left\{-\frac{[2\underline{s}^T\Sigma^{-1}\underline{s} + \underline{y}^T\Sigma^{-1}\underline{y}]}{2}\right\}}$$

$\mu_1)^2.$

$\underline{s}^T \Sigma^{-1} \underline{y}$

$\cancel{L(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'}$

$$\Rightarrow L(\underline{y}) = \frac{1}{2} \exp\left\{ \frac{\underline{y}^T \Sigma^{-1} \underline{y} - (\underline{y}-\underline{s})^T \Sigma^{-1} (\underline{y}-\underline{s})}{2} \right\} + \frac{1}{2} \exp\left\{ \frac{\underline{y}^T \Sigma^{-1} \underline{y} - (\underline{y}+\underline{s})^T \Sigma^{-1} (\underline{y}+\underline{s})}{2} \right.$$

$$= \frac{1}{2} \exp\left\{ \frac{2 \underline{s}^T \Sigma^{-1} \underline{y} - \underline{s}^T \Sigma^{-1} \underline{s}}{2} \right\} + \frac{1}{2} \exp\left\{ - \frac{2 \underline{s}^T \Sigma^{-1} \underline{y} + \underline{s}^T \Sigma^{-1} \underline{s}}{2} \right.$$

$$L(\underline{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma'$$

$$\Rightarrow \frac{1}{2} \exp\left\{ - \frac{\underline{s}^T \Sigma^{-1} \underline{s}}{2} \right\} \cdot \left[\exp\left\{ \underline{s}^T \Sigma^{-1} \underline{y} \right\} + \exp\left\{ - \underline{s}^T \Sigma^{-1} \underline{y} \right\} \right] \gtrless \gamma'$$

$$\Rightarrow e^{\underline{s}^T \Sigma^{-1} \underline{y}} + e^{-\underline{s}^T \Sigma^{-1} \underline{y}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma''$$

$$\Rightarrow \cosh\left(\underline{s}^T \Sigma^{-1} \underline{y} \right) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma''$$

$$\Rightarrow \boxed{\left| \underline{s}^T \Sigma^{-1} \underline{y} \right| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma}$$

$A C A^T$

$\underline{s}^T (\Sigma^{-1}) \Sigma (\Sigma^{-1})^T \underline{s}$

$\Sigma^{-1} \Sigma (\Sigma^{-1})^T$

$(\Sigma^{-1})^T$

$$\underline{y} \overset{H_0}{\sim} N(\underline{0}, \Sigma)$$
$$\overset{H_1}{\sim} \frac{1}{2} N(\underline{s}, \Sigma) + \frac{1}{2} N(-\underline{s}, \Sigma)$$

$$\Rightarrow \underline{s}^T \Sigma^{-1} \underline{y} \overset{H_0}{\sim} N\left(\underline{0}, \underline{s}^T (\Sigma^{-1})^T \underline{s}\right)$$
$$\overset{H_1}{\sim} \frac{1}{2} N\left(\underline{s}^T \Sigma^{-1} \underline{s}, \underline{s}^T (\Sigma^{-1})^T \underline{s}\right) + \frac{1}{2} N\left(-\underline{s}^T \Sigma^{-1} \underline{s}, \underline{s}^T (\Sigma^{-1})^T \underline{s}\right)$$

$$-\frac{(y+s)^T\Sigma^{-1}(y+s)}{2}\Big\}$$

$$\frac{\Sigma^{-1}y + s^T\Sigma^{-1}s}{2}\Big\}$$

$$\Big]\gtrless\gamma'$$

$$P_D = P\left(T(y) > \gamma/H_1\right) + P\left(T(y) < -\gamma/H_1\right)$$

$$= 2\,P\left(T(y) > \gamma/H_1\right)$$

$$P_D = Q\left(\frac{\gamma - s^T\Sigma^{-1}s}{\sqrt{s^T(\Sigma^{-1})^T s}}\right) + Q\left(\frac{\gamma + s^T\Sigma^{-1}s}{\sqrt{s^T(\Sigma^{-1})^T s}}\right)$$

$$P_{FA} = P\left(T(y) > \gamma/H_0\right) + P\left(T(y) < -\gamma/H_0\right)$$

$$= 2\,Q\left(\frac{\gamma}{\sqrt{s^T(\Sigma^{-1})^T s}}\right) = \alpha$$

$$\Rightarrow \gamma = \sqrt{s^T(\Sigma^{-1})^T s}\cdot Q^{-1}(\alpha/2)$$

$$\therefore\ P_D = Q\left(Q^{-1}\left(\frac{\alpha}{2}\right) - \frac{s^T(\Sigma^{-1})s}{\sqrt{s^T(\Sigma^{-1})^T s}}\right) + Q\left(Q^{-1}\left(\frac{\alpha}{2}\right) + \frac{s^T\Sigma^{-1}s}{\sqrt{s^T(\Sigma^{-1})^T s}}\right)$$

$$\cancel{\underline{y} \overset{H_1}{\sim} N\left(\underline{0}, \left(\sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2\right)\right)}$$

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ch3, prob7:-

(b) $H_0: Y_k = N_k, \quad k = 1, 2, \ldots, n$

versus $H_1: Y_k = N_k + \Theta S_k, \quad k = 1, 2, \ldots, n,$

where $\underline{N} \sim N(\underline{0}, C)$, $\bar{S}$ is known.

Given $\Theta \sim N(0, \sigma_\theta^2)$ & $C = \sigma^2 I$.

$Y_k \overset{H_1}{\sim} N(0, \sigma^2 + |s_k|^2 \sigma_\theta^2)$ & $Y_k \overset{H_0}{\sim} N(0, \sigma^2)$.

$$\Rightarrow \underline{y} \overset{H_1}{\sim} N\left(\underline{0}, \left(\sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2\right)\right), \quad \underline{y} \overset{H_0}{\sim} N(0, \sigma^2 I)$$

$$p_{\underline{y}/H_0}(\underline{y}/H_0) = \frac{1}{(2\pi)^{n/2} \cdot (\sigma^2)^{n/2}} \cdot \exp\left\{ - \frac{\underline{y}^T (\sigma^2 I)^{-1} \underline{y}}{2} \right\}.$$

$$p_{\underline{y}/H_1}(\underline{y}/H_1) = \frac{1}{(2\pi)^{n/2} \left| \sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2 \right|^{1/2}} \cdot \exp\left\{ - \frac{\underline{y}^T \left(\sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2 \right)^{-1} \underline{y}}{2} \right\}$$

$$\Rightarrow L(\underline{y}) = \frac{(\sigma^2)^{n/2}}{\left| \sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2 \right|^{1/2}} \cdot \exp\left\{ \frac{\underline{y}^T (\sigma^2 I)^{-1} \underline{y} - \underline{y}^T \left(\sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2 \right)^{-1} \underline{y}}{2} \right\} \quad \text{--- (1)}$$

Now, using the identity, $(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1}$,

where $\left(\sigma^2 I + \underline{s}\underline{s}^T \sigma_\theta^2\right)^{-1} = (\sigma^2 I)^{-1} - (\sigma^2 I)^{-1} \cdot \underline{s} \left[(\sigma_\theta^2 I)^{-1} + (\underline{s}^T \underline{s})(\sigma^2 I)^{-1} \right]^{-1} \underline{s}^T (\sigma^2 I)^{-1}$

$$= (\sigma^2 I)^{-1} - (\sigma^2 I)^{-1} \underline{s} \left[\frac{1}{\sigma_\theta^2} + \frac{\underline{s}^T \underline{s}}{\sigma^2} \right]^{-1} I \cdot \underline{s}^T \cdot \frac{I}{\sigma^2}$$

$$(\sigma^2 I)^{-1} \quad (\underline{s}\underline{s}^T) \cdot \frac{\sigma^2 \cdot \sigma_\theta^2}{\ldots} \cdot \frac{1}{\sigma^2} \cdot \frac{1}{\sigma^2}$$

$$= \left(\sigma^2 I\right)^{-1} - \frac{\sigma_\theta^2 \left(\underline{s}\,\underline{s}^T\right)}{\sigma^2\left[\sigma^2 + \underline{s}^T\underline{s}\,\sigma_\theta^2\right]} \quad \text{--- (2)}$$

Substituting (2) in (1) we can get,

$$\cancel{L(\underline{y}) = \frac{(\sigma^2)^{n/2}}{\left|\sigma^2 + \underline{s}^T\underline{s}\,\sigma_\theta^2\right|}}$$

$$L(\underline{y}) = \frac{(\sigma^2)^{n/2}}{\left|\sigma^2 I + \underline{s}^T\underline{s}\,\sigma_\theta^2\right|^{1/2}} \cdot \exp\left\{ \frac{\sigma_\theta^2}{2\sigma^2\left[\sigma^2 + \underline{s}^T\underline{s}\,\sigma_\theta^2\right]} \cdot \left(\underline{y}^T\underline{s}\right)\left(\underline{s}^T\underline{y}\right) \right\}$$

$$L(\underline{y}) = K_1 \cdot \exp\left\{ K_2 \left\| \underline{s}^T\underline{y} \right\|^2 \right\}$$

$$\therefore \quad K = \frac{\sigma_\theta^2}{2\sigma^2\left[\sigma^2 + \underline{s}^T\underline{s}\,\sigma_\theta^2\right]}.$$

This problems could also be solved using Bayesian approach.

Ch3, prob9:-

$$H_0 : \underline{y} = \underline{N}$$

versus

$$H_1 : \underline{y} = \underline{N} + A\left[(1-\theta)\,\underline{s}^{(0)} + \theta\,\underline{s}^{(1)}\right],$$

Where $\underline{N} \sim N(\underline{0}, I)$; $A > 0$;

$$P(\theta = 0) = P(\theta = 1) = 0.5$$

$\underline{s}^{(0)}, \underline{s}^{(1)}$ are orthogonal signals.

(a) $P_{\underline{y}/H_1}(\underline{y}/H_1) = P_{\underline{y}/H_1,\theta}(\underline{y}/H_1, \theta=0) \cdot P(\theta=0) + P_{\underline{y}/H_1,\theta_1}(\underline{y}/H_1, \theta=1) \cdot P(\theta=1)$

$$= 0.5 \times \frac{1}{(2\pi)^{n/2}} \cdot \exp\left\{ - \frac{[\underline{y} - \underline{s}^{(0)} A]^T [\underline{y} - \underline{s}^{(0)} A]}{2} \right\}$$

$$+ 0.5 \times \frac{1}{(2\pi)^{n/2}} \cdot \exp\left\{ - \frac{(\underline{y} - A\underline{s}^{(1)})^T [\underline{y} - A\underline{s}^{(1)}]}{2} \right\}$$

$$P_{\underline{y}/H_0}(\underline{y}/H_0) = \frac{1}{(2\pi)^{n/2}} \cdot \exp\left\{ - \frac{\underline{y}^T \underline{y}}{2} \right\}$$

$$\Rightarrow L(\underline{y}) = \frac{1}{2} \exp\left\{ \frac{\underline{y}^T\underline{y} - (\underline{y} - A\underline{s}^{(0)})^T (\underline{y} - A\underline{s}^{(0)})}{2} \right\} + \frac{1}{2} \exp\left\{ \frac{\underline{y}^T\underline{y} - (\underline{y} - A\underline{s}^{(1)})^T (\underline{y} - A\underline{s}^{(1)})}{2} \right\}$$

$$= \frac{1}{2} \exp\left\{ \frac{A[\underline{s}^{(0)T}\underline{y} + \underline{y}^T \underline{s}^{(0)}] - A^2}{2} \right\} + \frac{1}{2} \exp\left\{ \frac{A[\underline{s}^{(1)T}\underline{y} + \underline{y}^T \underline{s}^{(1)}] - A^2}{2} \right\} //$$

(b)

$H_0 : A = 0$

$H_1 : A > 0$

$$L(\underline{y}) = \frac{1}{2} \exp\left\{ -\frac{A^2}{2} \right\} \cdot \left[e^{A \underline{s}^{(0)T}\underline{y}} + A\, \underline{s}^{(1)T}\underline{y} \right]$$

LMP at $A_0 = 0$ is given by

$$\frac{\partial}{\partial A} L(\underline{y}) \Big|_{A=0} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \frac{1}{2} \cdot \exp\left\{ -\frac{A^2}{2} \right\} \cdot \left(-\frac{2A}{2} \right) \left[e^{A \underline{s}^{(0)T}\underline{y} + \underline{s}^{(1)T}\underline{y} A} \right] + \frac{1}{2} \exp\left\{ -\frac{A^2}{2} \right\} \cdot \left[e^{A \underline{s}^{(0)T}\underline{y} + A\underline{s}^{(1)T}\underline{y}} \right] \cdot \left(\underline{s}^{(0)T}\underline{y} + \underline{s}^{(1)T}\underline{y} \right) \Big|_{A=0} \gtrless \eta$$

$$\Rightarrow \underline{s}^{(0)T}\underline{y} + \underline{s}^{(1)T}\underline{y} \underset{H_0}{\overset{H_1}{\gtrless}} \eta .$$

$$\therefore \sum_{k=1}^{n} y_k \left[s_k^{(0)} + s_k^{(1)} \right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

(c) $T(\underline{y}) = \left\{ \underline{s}^{(0)} + \underline{s}^{(1)} \right\} \underline{y}$

Under H_0,

$$T(\underline{y}) \overset{H_0}{\sim} N\left(\underline{0}, \left[\underline{s}^{(0)} + \underline{s}^{(1)} \right]^T I \left[\underline{s}^{(0)} + \underline{s}^{(1)} \right] \right)$$

$$\overset{H_0}{\sim} N(\underline{0}, 2\mathbf{1}) \quad \text{, because } \underline{s}^{(0)}, \underline{s}^{(1)} \text{ are orthonormal.}$$

Under H_1,

$$T(\underline{y}) \overset{H_1}{\sim} P(\theta=0) \cdot N\left(A\left[\underline{s}^{(0)} + \underline{s}^{(1)} \right]^T \underline{s}^{(0)}, \left(\underline{s}^{(0)} + \underline{s}^{(1)} \right)^T I \left[\underline{s}^{(0)} + \underline{s}^{(1)} \right] \right)$$

$$+ P(\theta=1) \cdot N\left(A\left[\underline{s}^{(0)} + \underline{s}^{(1)} \right]^T \underline{s}^{(1)}, \left[\underline{s}^{(0)} + \underline{s}^{(1)} \right]^T I \left(\underline{s}^{(0)} + \underline{s}^{(1)} \right) \right)$$

$$\overset{H_1}{\sim} \frac{1}{2} N(\mathbf{1}A, 2\mathbf{1}) + \frac{1}{2} N(A\mathbf{1}, 2\mathbf{1})$$

$\underline{1} \rightarrow$ refers to $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_n$

$$\Rightarrow T(\underline{y}) \overset{H_1}{\sim} N(A\mathbf{1}, 2\mathbf{1})$$

$$\overset{H_0}{\sim} N(\underline{0}, 2\mathbf{1})$$

$$P_{FA} = P\{T(y) > \eta / H_0\} = Q\left(\frac{\eta}{\sqrt{2}}\right) = \alpha$$

$$\Rightarrow \eta = \sqrt{2}\, Q^{-1}(\alpha)$$

$$P_D = P\{T(y) > \eta / H_1\} = Q\left(\frac{\eta - A}{\sqrt{2}}\right) = Q\left(Q^{-1}(\alpha) - \frac{A}{\sqrt{2}}\right)$$

$$\therefore, \quad \underline{\underline{P_D = Q\left(Q^{-1}(P_{FA}) - \frac{A}{\sqrt{2}}\right)}}$$

<u>Ch3, prob 13:-</u>

$$Y_k = \theta^{1/2} s_k R_k + N_k, \quad k = 1, 2, \dots, n,$$

where $s_1, s_2, \dots, s_n$ is known signal sequence, $\theta \geq 0$ is constant.

$R_1, R_2, \dots, R_n, N_1, N_2, \dots, N_n \sim N(0, 1)$

(a) $H_0 : \theta = 0$

versus $H_1 : \theta = A$.

$Y_k \overset{H_0}{\sim} N(0, 1)$ & Y_k's are independent.

$\Rightarrow \underline{Y} \overset{H_0}{\sim} N(\underline{0}, I)$.

$Y_k \overset{H_1}{\sim} N(0, 1 + A s_k^2)$ & Y_k's are independent

$\Rightarrow \underline{Y} \overset{H_1}{\sim} N(\underline{0}, \underline{C})$, where $C = \mathrm{diag}(A s_1^2 + 1, \dots, A s_n^2 + 1)$

$$\Rightarrow L(\underline{y}) = \frac{P_{\underline{Y}/H_1}(\underline{y}/H_1)}{P_{\underline{Y}/H_0}(\underline{y}/H_0)} = \frac{1}{\left\{\prod_{k=1}^{n} (A s_k^2 + 1)\right\}^{1/2}} \exp\left\{\frac{1}{2} \sum_{k=1}^{n} \left(y_k^2 - \frac{y_k^2}{A s_k^2 + 1}\right)\right\}.$$

$$L(\underline{y}) = \frac{1}{\left[\prod_{k=1}^{n} (A s_k^2 + 1)\right]^{1/2}} \exp\left\{\frac{1}{2} \sum_{k=1}^{n} \frac{A s_k^2 y_k^2}{1 + A s_k^2}\right\}.$$

In this part, since $A, s_1, s_2 \dots, s_n$ are known,

$$\underbrace{\sum_{k=1}^{n} \frac{A s_k^2}{A s_k^2+1}\, y_k^2}_{T(\underline{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$T(\underline{y})$ is chi-square distributed.

(b)

$H_0 : \theta = 0$

$H_1 : \theta > 0$

From part (a),

$$L(\underline{y}) = \frac{1}{\left\{\prod_{k=1}^{n} (\theta s_k^2+1)\right\}^{1/2}} \exp\left\{ \frac{1}{2} \sum_{k=1}^{n} \frac{\theta s_k^2}{\theta s_k^2+1}\, y_k^2 \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma' \quad \text{--- (1)}$$

If all s_k's are of equal magnitudes, we have a test statistic independent of the actual value s_k's take & the +ve value θ takes.

~~Hence for UMP to exist,~~

If $|s_1| = |s_2| = \dots = |s_n|$, UMP exists.

(c) ~~LMP~~ Locally Most Powerful Test:-

$$\left.\frac{\partial L(\underline{y})}{\partial \theta}\right|_{\theta=0} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

① $\Rightarrow$

$$\sum_{K=1}^{n} \frac{\theta s_K^2}{\theta s_K^2+1} y_K^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma.$$

$$\underline{\underline{\text{LMP}}} \Rightarrow \sum_{K=1}^{n} \left. \frac{(\theta s_K^2+1) s_K^2 - \theta s_K^2 (s_K^2)}{(\theta s_K^2+1)^2} \right|_{\theta=0} y_K^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\Rightarrow \sum_{K=1}^{n} \frac{s_K^2}{1^2} y_K^2 \gtrless \gamma$$

$$\Rightarrow \boxed{\sum_{K=1}^{n} s_K^2 y_K^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma} \rightarrow \text{Locally Most Powerful test condition at } \theta_0 = 0.$$

But we need the value of s_K's to compute this.