2.2.1.

$H_1: \ r = s + n$

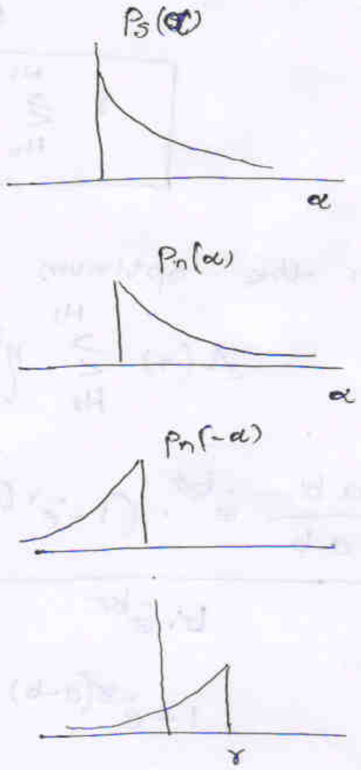
$H_0: \ r = n,$

$$P_s(s) = a e^{-as} \quad s \geq 0,$$
$$= 0 \quad s < 0,$$

$$P_n(N) = b e^{-bN} \quad N \geq 0,$$
$$N < 0.$$

s and n are independent random Variables.

(a)
$$f_{r/H_1}(r/H_1) = \int_{\alpha=0}^{r} p_s(\alpha) \cdot p_n(r-\alpha)\, d\alpha$$



$$= \int_{\alpha=0}^{r} a \cdot e^{-a\alpha} \cdot b \cdot e^{-b(r-\alpha)}\, d\alpha$$

$$= ab \cdot \int_{0}^{r} e^{-a\alpha} \cdot e^{-b(r-\alpha)}\, d\alpha$$

$$= ab \cdot e^{-br} \int_{0}^{r} e^{-\alpha(a-b)}\, d\alpha$$

$$= ab \cdot e^{-br} \cdot \left. \frac{e^{-\alpha(a-b)}}{-(a-b)} \right]_0^{r}$$

$$= \frac{ab}{a-b} \cdot e^{-br} \left[1 - e^{-r(a-b)} \right], \quad r \geq 0.$$

$P_s(\alpha) \cdot P_n(r-\alpha)$ is non zero only for $0 \leq \alpha \leq r$.

$$f_{r/H_0}(r/H_0) = b \cdot e^{-br}, \quad r \geq 0$$

$$\Lambda(r) = \frac{f_{r/H_1}(r/H_1)}{f_{r/H_0}(r/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta'$$

$$\frac{\frac{ab}{a-b} \cdot e^{-br} \cdot \left(1 - e^{-r(a-b)}\right)}{b \cdot e^{-br}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta'$$

$$1 - e^{-r(a-b)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta''$$

$$e^{-r(a-b)} \underset{H_0}{\overset{H_1}{\lessgtr}} \eta'' - 1$$

by taking log on both Sides

$$-r(a-b) \underset{H_0}{\overset{H_1}{\lessgtr}} \log_e(\eta''-1)$$

$$\boxed{r \underset{H_0}{\overset{H_1}{\gtrless}} \eta}$$

(b) for the optimum Bayes test find η ?

$$\Lambda(r) \underset{H_0}{\overset{H_1}{\gtrless}} \eta' = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

$$\frac{\frac{ab}{a-b} \, e^{-br} \cdot \left(1 - e^{-r(a-b)}\right)}{b \cdot e^{-br}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta'$$

$$1 - e^{-r(a-b)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta' \left(\frac{a-b}{a}\right)$$

$$e^{-r(a-b)} \underset{H_0}{\overset{H_1}{\lessgtr}} \left(-\eta' \left(\frac{a-b}{a}\right) + 1\right)$$

by taking log on both. Sides,

$$-r(a-b) \underset{H_0}{\overset{H_1}{\lessgtr}} \log\left(-\eta' \left(\frac{a-b}{a}\right) + 1\right)$$

$$r \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log\left(-\eta'\left(\frac{a-b}{a}\right) + 1\right)}{-(a-b)} = \eta$$

(c) $P_F \triangleq Pr(\text{say } H_1 / H_0 \text{ is true}) = \int f_{\Lambda/H_0}(\Lambda/H_0)\, d\Lambda$

$$P_F = \int_{\eta}^{\infty} b \cdot e^{-br} \cdot dr$$

$$= b \cdot \frac{e^{-br}}{-b}\Big]_{\eta}^{\infty}$$

$$= -[0 - e^{-b\eta}] = e^{-b\eta}$$

$$P_F = e^{-b\eta} \Rightarrow \log P_F = -b\eta$$

$$\Rightarrow \eta = \frac{\log(P_F)}{(-b)}$$

2.2.18 $f_{X/H_1}(X/H_1) = \sum_{K=1}^{M} P_K \frac{1}{(2\pi\sigma^2)^{M/2}} e^{-\left[\frac{(X_K-m)^2}{2\sigma^2}\right]} \prod_{i\neq K}^{M} e^{\left(-\frac{X_i^2}{2\sigma^2}\right)}$

$$\sum_{k=1}^{M} P_K = 1,$$

$$f_{X/H_0}(X/H_0) = \prod_{i=1}^{M} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{X_i^2}{2\sigma^2}\right)} \qquad -\infty < X_i < \infty.$$

$$\Lambda(x) = \frac{f_{X/H_1}(x/H_1)}{f_{X|H_0}(x/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\sum_{K=1}^{M} \frac{P_K \cdot 1 e^{-\left[\frac{X_K^2}{2\sigma^2}\right]}}{(2\pi\sigma^2)^{M/2}} \cdot e^{-\left[\frac{m^2-2mX_K}{2\sigma^2}\right]} \prod_{i\neq K}^{M} \exp\left[\frac{-X_i^2}{2\sigma^2}\right]}{\prod_{i=1}^{M} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{X_i^2}{2\sigma^2}\right)}} \underset{H_1}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\prod_{i=1}^{M} \frac{1 e^{-\left[\frac{X_i^2}{2\sigma^2}\right]}}{\sqrt{2\pi}\sigma} \sum_{K=1}^{M} P_K e^{-\left[\frac{m^2-2mX_K}{2\sigma^2}\right]}}{\prod_{i=1}^{M} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left[\frac{X_i^2}{2\sigma^2}\right]}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$e^{-\frac{m^2}{2\sigma^2}} \sum_{k=1}^{M} p_k \cdot e^{\frac{m x_k}{\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta.$$

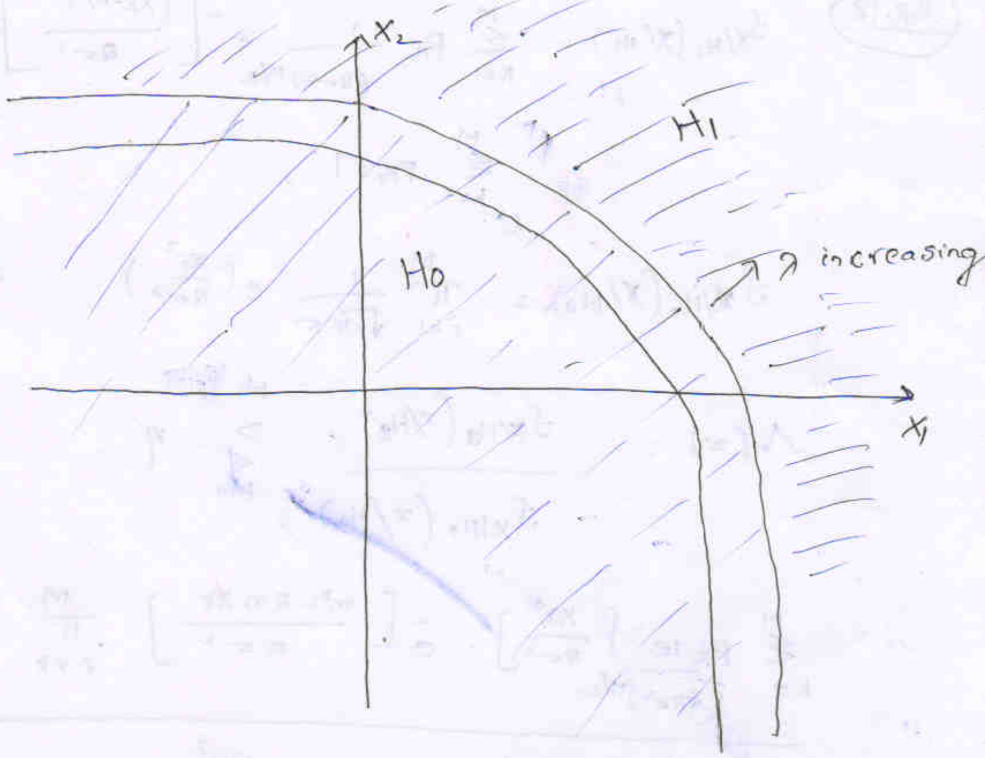
$$\sum_{k=1}^{M} p_k \cdot e^{\frac{m x_k}{\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta'$$

(b) $M = 2$, and $p_1 = p_2 = 1/2$

$$\frac{1}{2}\left[e^{\frac{m x_1}{\sigma^2}} + e^{\frac{m x_2}{\sigma^2}} \right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta'$$

$$\Rightarrow \quad e^{\frac{m x_1}{\sigma^2}} + e^{\frac{m x_2}{\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} 2\eta'$$

$$\Rightarrow \quad e^{k \cancel{x_1}} + e^{k x_2} \underset{H_0}{\overset{H_1}{\gtrless}} \eta''' = \cancel{\lambda} = \lambda$$



1(c) $e^{kx_1} + e^{kx_2} \underset{H_0}{\overset{H_1}{\gtrless}} \eta_1''' = \lambda$

let $x_1 >> x_2$

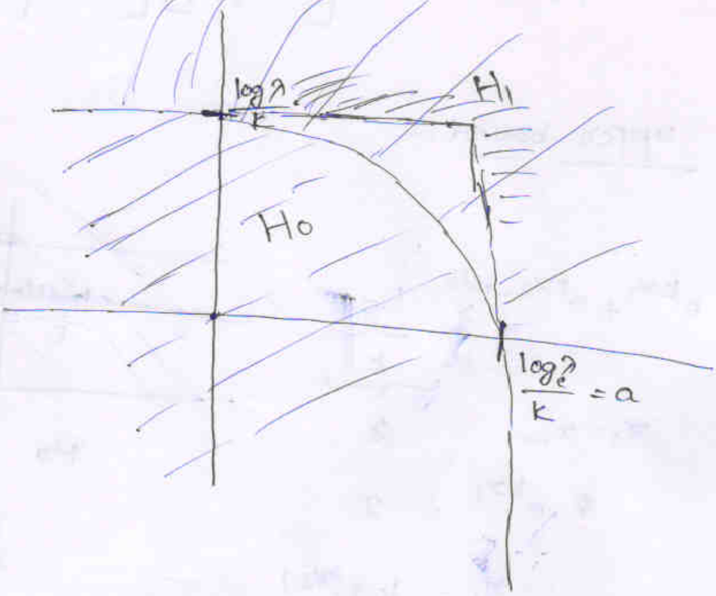
$e^{kx_1} \underset{H_0}{\overset{H_1}{\gtrless}} \eta''' = \lambda$

$k x_1 \underset{H_0}{\overset{H_1}{\gtrless}} \log_e \lambda$

$x_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log_e \lambda}{k}$

Similarly $x_2 >> x_1$

$x_2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log_e \lambda}{k}$



lower bound;

$$P_{FA} > P(\Lambda(x) > \lambda / H_0)$$

$$= 1 - \int_{-\infty}^{a}\int_{-\infty}^{a} p_{x_1 x_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= 1 - \int_{-\infty}^{a} p_{x_1}(x_1)\, dx_1 \int_{-\infty}^{a} p_{x_2}(x_2)\, dx_2$$

$$= 1 - \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_1^2}{2\sigma^2}}\, dx_1 \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_2^2}{2\sigma^2}}\, dx_2$$

$$= 1 - \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2$$

$$P_D > P(\Lambda(x) > \lambda / H_1)$$

$$= 1 - \int_{-\infty}^{a}\int_{-\infty}^{a} \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_1-m)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x_2^2}{2\sigma^2}\right)\right]$$

$$+ \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_2-m)^2}{2\sigma^2}\right] \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x_1^2}{2\sigma^2}\right)\right] dx_1 dx_2$$

$$P_D \geq 1 - \frac{1}{2}\left[\left(1 - Q\left(\frac{a}{\sigma}\right)\right]\left[1 - Q\left(\frac{a-m}{\sigma}\right)\right] + \left(1 - Q\left(\frac{a}{\sigma}\right)\right)\left(1 - Q\left(\frac{a-m}{\sigma}\right)\right)\right]$$

$$P_D > 1 - \left[1 - Q\left(\frac{a}{\sigma}\right)\right]\left[1 - Q\left(\frac{a-m}{\sigma}\right)\right]$$

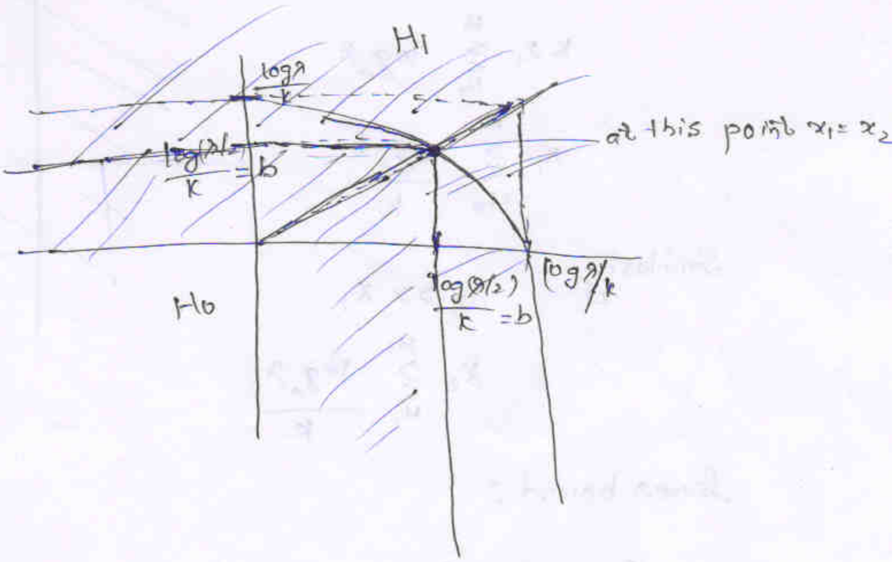
upper bound:-

$$e^{kx_1} + e^{kx_2} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{\log \lambda}{k}}_{\lambda}$$

$x_1 = x_2$

$\& \quad e^{kx_1} = \lambda$

$$x_1 = \frac{\log(\lambda/2)}{k}$$



$$P_{FA} < 1 - \int_{-\infty}^{b}\int_{-\infty}^{b} P_{x_1 x_2}(x_1, x_2)\, dx_1\, dx_2$$

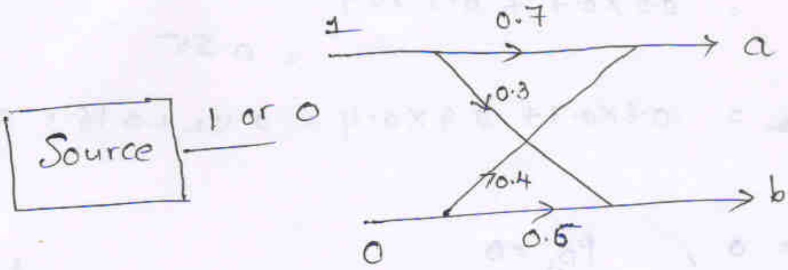
$$= 1 - \left[1 - Q\left(\frac{b}{\sigma}\right)\right]^2$$

$$P_D < 1 - \left[\left(1 - Q\left(\frac{b}{\sigma}\right)\right]\left(1 - Q\left(\frac{b-m}{\sigma}\right)\right)\right]$$

2.27:

S_1 : $Pr(1) = 0.5$, $Pr(0) = 0.5$;

S_2 : $Pr(1) = 0.6$, $Pr(0) = 0.4$.



the four possible cases are,

$r = a \rightarrow$ decide S_1
$r = b \rightarrow$ decide S_2 } (1)

$r = a \rightarrow$ decide S_2
$r = b \rightarrow$ decide S_1 } (2)

$r = a \rightarrow$ decide S_1
$r = b \rightarrow$ decide S_1 } (3)

$r = a \rightarrow$ decide S_2
$r = b \rightarrow$ decide S_2 } (4)

✓ $P_{FA_1} = P(\text{Decide } S_2 / S_1 \text{ is true})$

$= Pr(r = b / S_1 \text{ is true})$

$= 0.5 \times 0.3 + 0.5 \times 0.5$

$= 0.15 + 0.30 = 0.45$

$P_{D_1} = P(\text{decide } S_2 / S_2 \text{ is true})$

$= Pr(r = b / S_2 \text{ is true})$

$= 0.6 \times 0.3 + 0.4 \times 0.6 = 0.18 + 0.24 = 0.42$

$$P_{FA_2} = P\left(\text{decide } s_2 / s_1 \text{ true}\right)$$

$$= P\left(r = a / s_1 \text{ is true}\right)$$

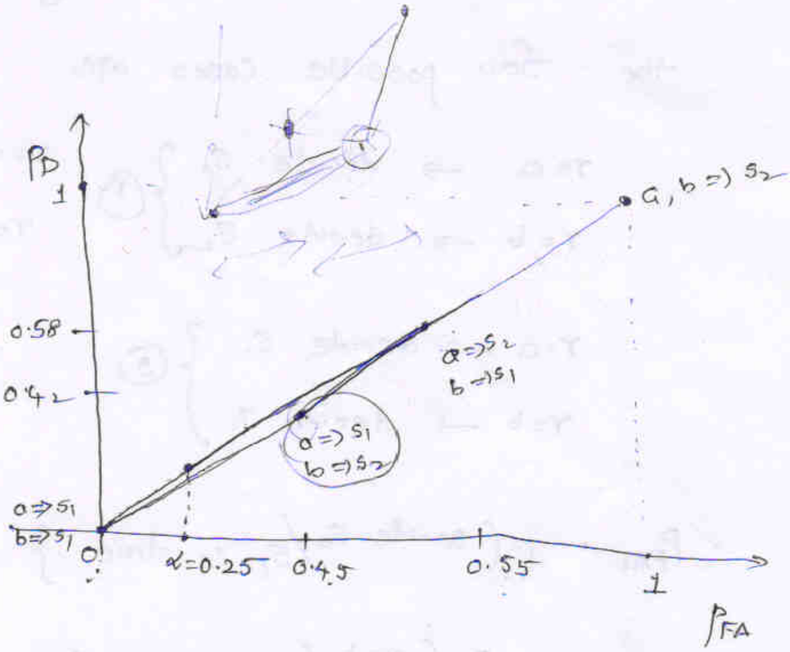
$$= 0.5 \times 0.7 + 0.5 \times 0.4 = 0.35 + 0.20$$

$$= 0.55$$

$$P_{DA_2} = 0.6 \times 0.7 + 0.4 \times 0.4 = 0.42 + 0.16 = 0.58$$

$P_{FA_3} = 0$, $\quad P_{D_3} = 0$

$P_{FA_4} = 1$, $\quad P_{D_4} = 1$



for $\alpha = 0.25$

take condition ② & condition ③

$b \Rightarrow s_1$ is decide

$a \Rightarrow s_1$ with prob p

$\quad s_2$ with prob $(1-p)$

$$0.25 = P_{FA_3} \times p + P_{FA_2} \times (1-p)$$

$$= 0 \times p + 0.55 \times (1-p)$$

$$1 - p = \frac{0.25}{0.55} = \frac{5}{11}$$

$$p = 6/11.$$

2.3.7.
$$H_0 : r = m_0 + n,$$
$$H_1 : r = m_1 + n,$$
$$H_2 = m_2 + n,$$

$$r \triangleq \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad m_i \triangleq \begin{bmatrix} m_{i1} \\ m_{i2} \\ m_{i3} \end{bmatrix} \quad n \triangleq \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

the components of n are statistically independent,

$$\Lambda_1 = \frac{f_{r/H_1}}{f_{r/H_0}} \qquad r \overset{H_1}{\sim} N(m_1, \sigma^2 I_{3\times3})$$
$$\overset{H_0}{\sim} N(m_0, \sigma^2 I)$$
$$\overset{H_2}{\sim} N(m_2, \sigma^2 I)$$

$$\Lambda_1 = \frac{\frac{1}{(\sqrt{2\pi\sigma^2})^3} \exp\left(-\sum_{i=1}^{3} \frac{(r_i - m_{1i})^2}{2\sigma^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma^2})^3} \exp\left(-\sum_{i=1}^{3} \frac{(r_i - m_{0i})^2}{2\sigma^2}\right)}$$

$$= \exp\left(\sum_{i=1}^{3} \frac{(\cancel{r_i^2} + m_{0i}^2 - 2m_{0i}r_i) - (\cancel{r_i^2} + m_{1i}^2 - 2m_{1i}r_i)}{2\sigma^2}\right)$$

$$l_1 \triangleq \log(\Lambda_1(r))$$
$$= \sum_{i=1}^{3} r_i (m_{1i} - m_{0i}).$$

$$l_2 \triangleq \sum_{i=1}^{3} r_i (m_{2i} - m_{0i})$$

(b) $\quad C_{00} = C_{11} = C_{22} = 0$

$$C_{12} = C_{21} = C_{01} = C_{10} = \frac{1}{2} C_{02} = \frac{1}{2} C_{20} > 0$$

$$\pi_1 (C_{01} - C_{11}) \Lambda_1 \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{>}} \pi_0 (C_{10} - C_{00}) + \pi_2 (C_{12} - C_{02}) \Lambda_2$$

$$\pi_1 \cdot C_{01} \Lambda_1 \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \pi_0 \cdot C_{10} + \pi_2 (C_{01} - 2C_{01}) \Lambda_2$$

$$\pi_1 \cdot \Lambda_1 \gtrless \pi_0 - \pi_2 \Lambda_2 \quad \text{——} \ (1)$$

$$\pi_2 (C_{02} - C_{22}) \Lambda_2 \underset{H_0 \text{ or } H_1}{\overset{H_2 \text{ or } H_1}{\gtrless}} \pi_0 (C_{20} - C_{00}) + \pi_1 (C_{21} - C_{01}) \Lambda_1$$

$$\pi_2 \cdot 2C_{01} \Lambda_2 \gtrless \pi_0 \cdot 2C_{01} + \pi_1 (0) \cdot \Lambda_1$$

$$\pi_2 \Lambda_2 \gtrless \pi_0 \quad \text{——} \ (2)$$

$$\pi_2 (C_{12} - C_{22}) \Lambda_2 \underset{H_1 \text{ or } H_0}{\overset{H_2 \text{ or } H_0}{\gtrless}} \pi_0 (C_{20} - C_{10}) + \pi_1 (C_{21} - C_{11}) \Lambda_1$$

$$\pi_2 \cdot C_{12} \Lambda_2 \gtrless \pi_0 \cdot C_{10} + \pi_1 \cdot C_{21} \cdot \Lambda_1$$

$$\pi_2 \Lambda_2 \gtrless \pi_0 + \pi_1 \cdot \Lambda_1 \quad \text{——} \ (3)$$

$$\pi_1 \cdot \Lambda_1 + \pi_2 \Lambda_2 \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \pi_0 \quad \text{——} \ (i)$$

$$\Lambda_2 \underset{H_0 \text{ or } H_1}{\overset{H_2 \text{ or } H_1}{\gtrless}} \pi_0 / \pi_2 \quad \text{——} \ (ii)$$

$$-\pi_1 \Lambda_1 + \pi_2 \Lambda_2 \underset{H_1 \text{ or } H_0}{\overset{H_2 \text{ or } H_0}{\gtrless}} \pi_0 \quad \text{——} \ (iii)$$

$$\log(\Lambda_1) = l_1 + \lambda_1 \quad \Rightarrow \quad \Lambda_1 = e^{l_1} * e^{\lambda_1}$$

$$\lambda_1 = \sum_{i=1}^{3} (m_{0i}^2 - m_{1i}^2)$$

$$\log(\Lambda_2) = l_2 + \lambda_2$$

$$\lambda_2 = \sum_{i=1}^{3} (m_{0i}^2 - m_{1i}^2)$$

$$\pi_1 \cdot \Lambda_1 + \pi_2 \cdot \Lambda_2 \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \pi_0$$

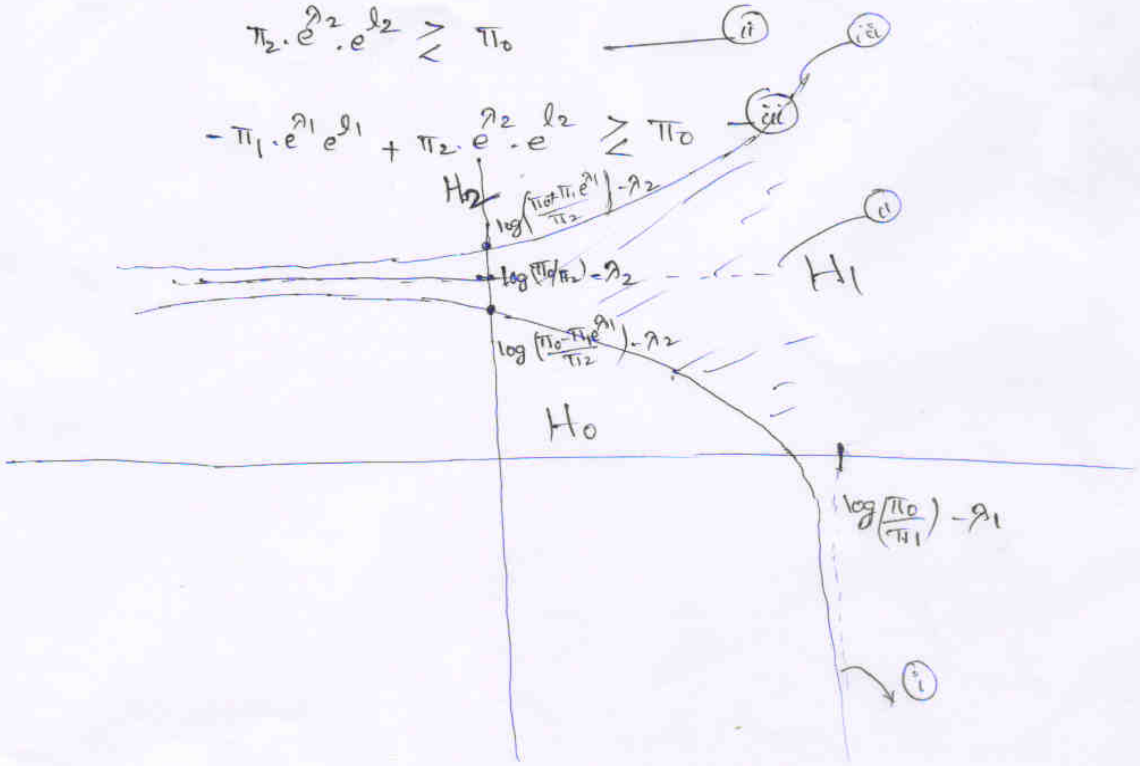
$$\pi_2 \cdot \Lambda_2 \underset{H_0 \text{ or } H_1}{\overset{H_2 \text{ or } H_1}{\gtrless}} \pi_0$$

$$-\pi_1 \Lambda_1 + \pi_2 \Lambda_2 \underset{H_1 \text{ or } H_0}{\overset{H_2 \text{ or } H_0}{\gtrless}} \pi_0$$

$$\pi_1 \cdot e^{\lambda_1} e^{l_1} + \pi_2 \cdot e^{\lambda_2} \cdot e^{l_2} \underset{H_0 \text{ or } H_2}{\overset{H_1 \text{ or } H_2}{\gtrless}} \pi_0 \quad \text{— (i)}$$

$$\pi_2 \cdot e^{\lambda_2} \cdot e^{l_2} \gtrless \pi_0 \quad \text{— (ii)}$$

$$-\pi_1 \cdot e^{\lambda_1} e^{l_1} + \pi_2 \cdot e^{\lambda_2} \cdot e^{l_2} \gtrless \pi_0 \quad \text{— (iii)}$$



2.2.6.

$H_1 : r = b m_1 + n$

$H_0 : r = n$

b and n are independent Zero-mean Gaussian Variables with Variances σ_b^2 and σ_n^2

$b \sim \mathcal{N}(0, \sigma_b^2)$

$n \sim \mathcal{N}(0, \sigma_n^2)$

$$f_{r/H_1}(r/H_1) \sim \mathcal{N}(0, \underbrace{m_1^2\sigma_b^2 + \sigma_n^2}_{\sigma_{r_1}^2})$$

$$f_{r/H_0}(r/H_0) \sim \mathcal{N}(0, \sigma_n^2)$$

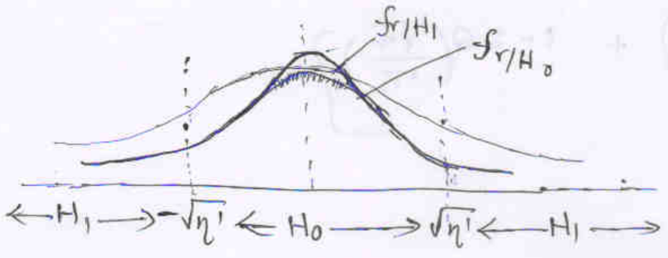
$$\frac{f_{r/H_1}(r/H_1)}{f_{r/H_0}(r/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\frac{1}{\sqrt{2\pi}\,\sigma_{r_1}} e^{-\frac{r^2}{2\sigma_{r_1}^2}}}{\frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{r^2}{2\sigma_n^2}}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\sigma_n}{\sigma_{r_1}} e^{\frac{1}{2}\left[\frac{1}{\sigma_n^2} - \frac{1}{\sigma_{r_1}^2}\right] r^2} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow r^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta' \qquad (\because \sigma_{n_1}^2 > \sigma_n^2)$$

$$\Rightarrow |r| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\eta'}$$



$$P_F = \int_{\sqrt{\eta'}}^{\infty} f_{\Lambda/H_0}\, d\Lambda + \int_{-\infty}^{-\sqrt{\eta'}} f_{\Lambda/H_0}\, d\Lambda$$

$$= \int_{\sqrt{\eta'}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{r^2}{2\sigma_\eta^2}}\, dr + \int_{-\infty}^{-\sqrt{\eta'}}$$

$$= Q\left(\frac{\sqrt{\eta'}}{\sigma_n}\right) + Q\left(\frac{\sqrt{\eta'}}{\sigma_n}\right)$$

$$= 2\,Q\left(\frac{\sqrt{\eta'}}{\sigma_n}\right) \Rightarrow \sqrt{\eta'} = \sigma_\eta\, Q^{-1}(P_F/2)$$

$$P_D = 2\,Q\left(\frac{\sqrt{\eta'}}{\sigma_{r_1}}\right)$$

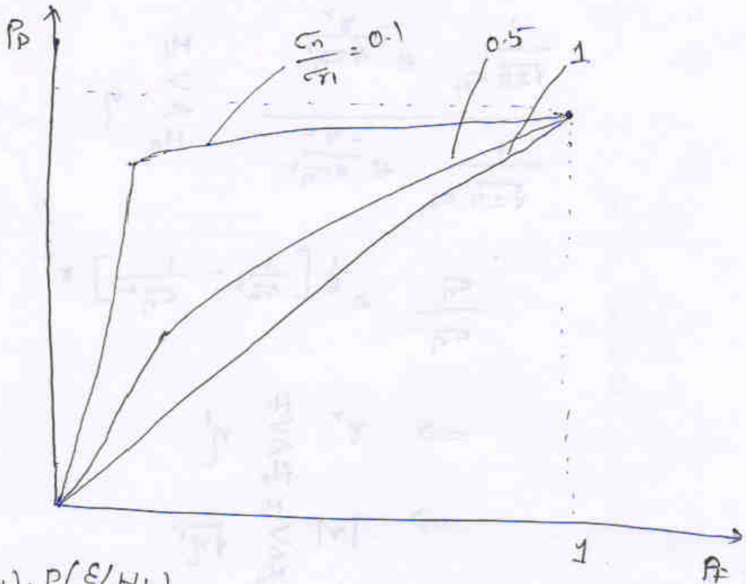
$$= 2\,Q\left(\frac{\sigma_n}{\sigma_{r_1}} Q^{-1}\left(\frac{P_F}{2}\right)\right)$$

(c) two hypothesis are equally likely,

use ML detector,

$$e^{\frac{1}{2}\left[\frac{1}{\sigma_\eta^2} - \frac{1}{\sigma_{r_1}^2}\right] r^2} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma_{r_1}}{\sigma_\eta}$$

$$r^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log\left(\frac{\sigma_{r_1}}{\sigma_n}\right)}{\frac{1}{2}\left[\frac{1}{\sigma_\eta^2} - \frac{1}{\sigma_{r_1}^2}\right]} = \eta$$



$$P(\mathcal{E}) = P(H_0)\cdot P(\mathcal{E}/H_0) + P(H_1)\cdot P(\mathcal{E}/H_1)$$

$$= \frac{1}{2}\left[2\,Q\left(\frac{\sqrt{\eta}}{\sigma_n}\right) + 1 - 2Q\left(\frac{\sqrt{\eta}}{\sigma_{r_1}}\right)\right]$$

2.2.13

$$\Lambda(R) = \frac{f_{r/H_1}(r/H_1)}{f_{r/H_0}(r/H_0)}$$

1. $$E[\Lambda^{n+1}/H_0] = \int \Lambda^{n+1} f_{r/H_0}(r/H_0)\, dr$$

$$= \int \Lambda^{n} \cdot \frac{f_{r/H_1}(r/H_1)}{\cancel{f_{r/H_0}(r/H_0)}} \cdot \cancel{f_{r/H_0}(r/H_0)}\, dr$$

$$= E[\Lambda^{n}/H_1]$$

2. $$E[\Lambda/H_0] = E[\Lambda^0/H_1] = E[1/H_1] = 1.$$

3. $$\mathrm{Var}[\Lambda/H_0] = E[\Lambda^2/H_0] - \{E[\Lambda/H_0]\}^2$$

$$= E[\Lambda/H_1] - 1^2$$

$$= E[\Lambda/H_1] - E[\Lambda/H_0]$$

2.2.19

$$f_{r_i/H_k}(R_i/H_k) = \frac{1}{\sqrt{2\pi}\,\sigma_k} \exp\left[-\frac{(R_i - m_k)^2}{2\sigma_k^2}\right] \quad i = 1, 2, \cdots N$$

$$k = 0, 1.$$

$$\Lambda(r) = \frac{f_{\underline{r}/H_1}(\underline{r}/H_1)}{f_{\underline{r}/H_0}(\underline{r}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\left(\frac{1}{\sqrt{2\pi}\,\sigma_1}\right)^N \cdot \exp\left[-\sum_{i=1}^{N} \frac{(R_i - m_1)^2}{2\sigma_1^2}\right]}{\left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^N \cdot \exp\left[-\sum_{i=1}^{N} \frac{(R_i - m_0)^2}{2\sigma_0^2}\right]} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\left(\frac{\sigma_0}{\sigma_1}\right)^N \exp\left[\sum_{i=1}^{N}\frac{(R_i-m_0)^2}{2\sigma_0^2}-\frac{(R_i-m_1)^2}{2\sigma_1^2}\right]\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\sum_{i=1}^{N}\left\{R_i^2\left[\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right]+R_i\left[\frac{m_1}{\sigma_1^2}-\frac{m_0}{\sigma_0^2}\right]\right\}\underset{H_0}{\overset{H_1}{\gtrless}}\eta'$$

let $l_\alpha = \sum_{i=1}^{N} R_i,$

$$l_\beta = \sum_{i=1}^{N} R_i^2$$

$$l_\beta\cdot\left[\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right]+l_\alpha\left[\frac{m_1}{\sigma_1^2}-\frac{m_0}{\sigma_0^2}\right]\underset{H_0}{\overset{H_1}{\gtrless}}\eta'$$

(b) decision regions in l_α, l_β-plane for

$$2m_0 = m_1 > 0$$

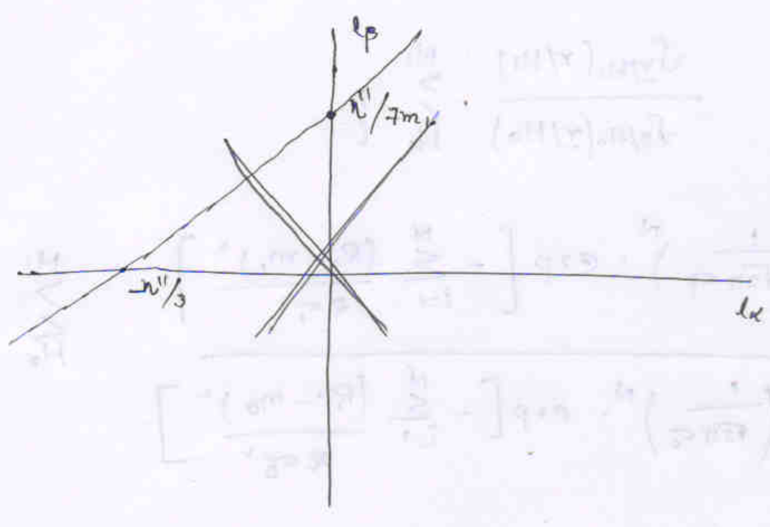
$$2\sigma_1 = \sigma_0$$

$$l_\beta\cdot\left[\frac{1}{8\sigma_1^2}-\frac{1}{2\sigma_1^2}\right]+l_\alpha\left[\frac{m_1}{\sigma_1^2}-\frac{m_1/2}{4\sigma_1^2}\right]\underset{H_0}{\overset{H_1}{\gtrless}}\eta'$$

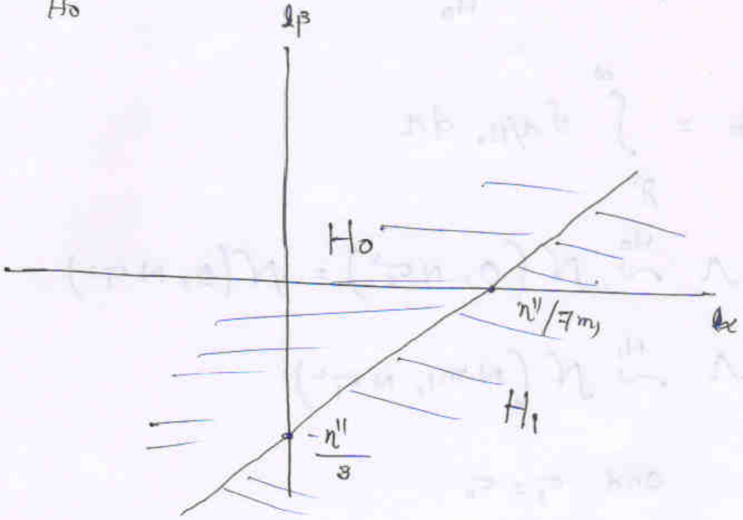
$$l_\beta\cdot\frac{-3}{8\sigma_1^2}+l_\alpha\cdot\frac{7m_1}{8\sigma_1^2}\underset{H_0}{\overset{H_1}{\gtrless}}\eta'$$

$$(-3)\,l_\beta+(7m_1)\,l_\alpha\underset{H_0}{\overset{H_1}{\gtrless}}\eta''$$

$(-3)\,l_\beta+(7m_1)\,l_\alpha=\eta''$



$$(7m_1)\, l_\alpha + (-3)\, l_\beta \underset{H_0}{\overset{H_1}{\gtrless}} \eta'' = 8\sigma_1^2 \eta'$$



2.2.20

1. $m_0 = 0$

$\sigma_0 = \sigma_1$

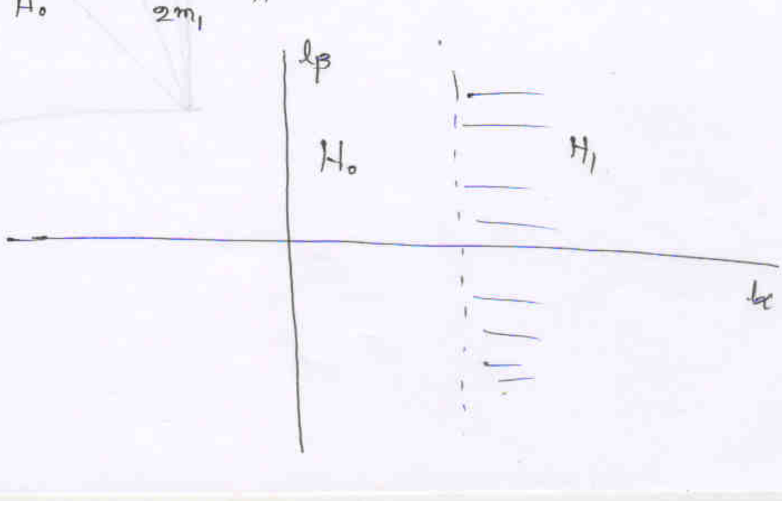
decision regions and ROC.

$$2\, l_\alpha \left[\frac{m_1}{\sigma_1^2} - \frac{m_0}{\sigma_0^2} \right] + l_\beta \left[\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right] \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$m_0 = 0, \quad \sigma_0 = \sigma_1$$

$$2\, l_\alpha \cdot \frac{m_1}{\sigma_1^2} + l_\beta\,(0) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$l_\alpha \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \cdot \frac{\sigma_1^2}{2m_1} = \lambda'$$



$$\Lambda = \sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \lambda'$$

$$P_F = \int_{\lambda'}^{\infty} f_{\Lambda/H_0}\, d\Lambda$$

$$\Lambda \overset{H_0}{\sim} \mathcal{N}(0, N\sigma_0^2) = \mathcal{N}(0, N\sigma_1^2)$$

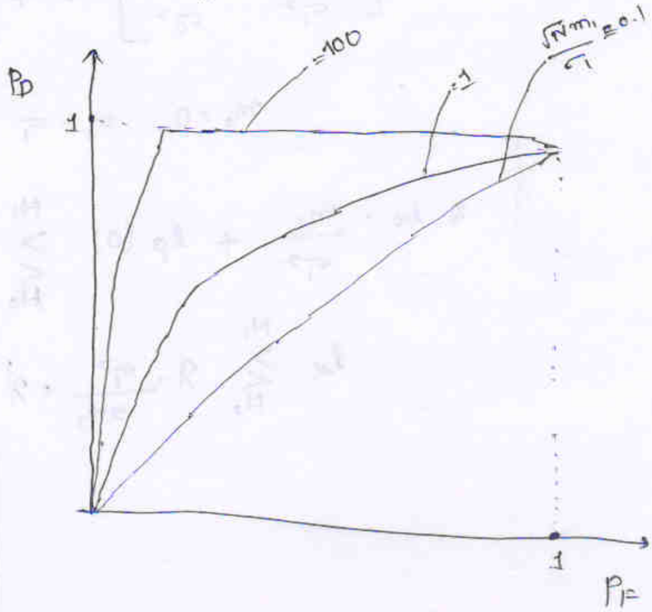
$$\Lambda \overset{H_1}{\sim} \mathcal{N}(N m_1, N\sigma_1^2)$$

and $\sigma_1 = \sigma_0$

$$P_F = Q\left(\frac{\lambda'}{\sqrt{N}\sigma_1}\right) \Rightarrow \frac{\lambda'}{\sqrt{N}\sigma_1} = Q^{-1}(P_F)$$

$$P_D = Q\left(\frac{\lambda' - N m_1}{\sqrt{N}\sigma_1}\right)$$

$$= Q\left(\frac{\lambda'}{\sqrt{N}\sigma_1} - \frac{\sqrt{N} m_1}{\sigma_1}\right) = Q\left(Q^{-1}(P_F) - \frac{\sqrt{N} m_1}{\sigma_1}\right)$$



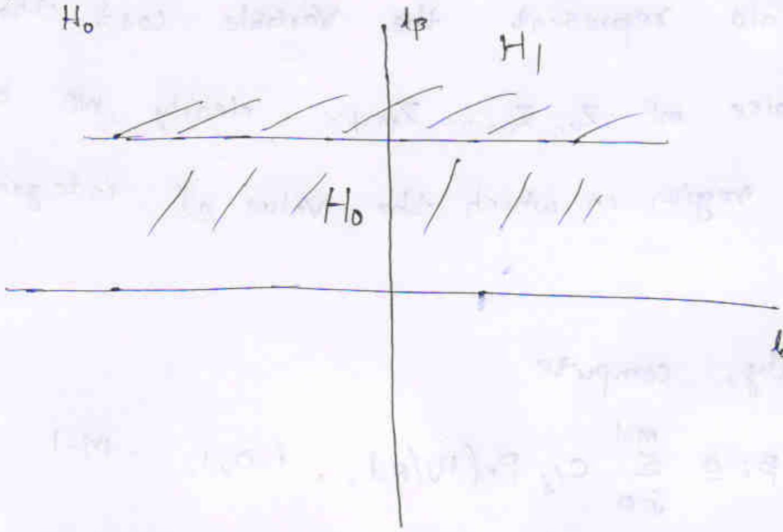
2. $m_0 = m_1 = 0$

$\sigma_1^2 = \sigma_s^2 + \sigma_n^2$

$\sigma_0 = \sigma_n$

$$l_\beta \cdot \left[\frac{1}{\sigma_n^2} - \frac{1}{\sigma_s^2 + \sigma_n^2} \right] \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$l_\beta \underset{H_0}{\overset{H_1}{\gtrless}} \lambda''$$



2.3.2

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} P_j C_{ij} \int_{Z_i} P_{r/H_j}(R/H_j)\, dR \qquad \text{Here, } P_{r/H_j} = f_{r/H_j}$$

$$= \sum_{i=0}^{M-1} \int_{Z_i} \sum_{j=0}^{M-1} C_{ij}\, P_j\, P_{r/H_j}(R/H_j)\, dR$$

devide by $P_r(r)$

$$= \sum_{i=0}^{M-1} \int_{Z_i} \sum_{j=0}^{M-1} C_{ij} \frac{P_j\, P_{r/H_j}(R/H_j)}{P_r(r)}\, dR$$

$$= \sum_{i=0}^{M-1} \int_{Z_i} \sum_{j=0}^{M-1} c_{ij} \, P_{H_j/r}(H_j/R) \, dR$$

$$= \int_{Z_0} \sum_{j=0}^{M-1} c_{0j} \, P_{H_j/r}(H_j/R) \, dR + \cdots\cdots$$

$$\cdots + \int_{Z_{M-1}} \sum_{j=0}^{M-1} c_{M-1\,j} \, P_{H_j/r}(H_j/R) \, dR$$

the integrals represent the variable cost that depends on our choice of $Z_0, Z_1 \ldots Z_{M-1}$. clearly, we assign each R to the region in which the value of integrand is the smallest.

So clearly, compute

$$\beta_i \triangleq \sum_{j=0}^{M-1} c_{ij} \Pr(H_j/R), \quad i = 0, 1, \ldots M-1$$

and Bayes test is choose the smallest.

(b) $C_{ii} = 0, \quad i = 0, 1, \ldots M-1$

$c_{ij} = c \quad i \neq j, \; i, j = 0, 1, \ldots M-1$

$$\beta_i = \sum_{j=0}^{M-1} c_{ij} \Pr(H_j/R)$$

$$= \sum_{\substack{j=0 \\ j \neq i}}^{M-1} \underbrace{c_{ij}}_{c} \Pr(H_j/R)$$

$$= \sum_{\substack{j=0 \\ j \neq i}}^{M-1} c \Pr(H_j/R)$$

and we know $\sum_{j=0}^{M-1} P(H_j/R) = 1$

decision rule is : $\arg\min_i \beta_i$

$$= \arg\min_i \sum_{\substack{j=0 \\ j\neq i}}^{M-1} P_r(H_i/R)$$

$$= \arg\min_i \left(1 - P(H_i/R)\right)$$

$$= \arg\max_i P(H_i/R)$$

finally compute $P(H_i/R)$ $i = 0, 1, 2, \ldots M-1$

choose largest.

2.3-3.

$$f_{r/H_k}(r/H_k) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(R-m_k)^2}{2\sigma^2}\right) \qquad -\infty < R < \infty$$

$$k = 1, 2, \ldots 5$$

$m_1 = -2m$
$m_2 = -m$
$m_3 = 0$
$m_4 = m$
$m_5 = 2m.$

Since prior probabilities are equal, ML detector is the optimal detector

Applying ML-detection, the detected index d is

$$d = \arg\max_i f(r/H_i)$$

$$= \arg\max_i \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(R-m_k)^2}{2\sigma^2}\right)$$

$$= \arg\min_i \frac{(R-m_k)^2}{2\sigma^2}$$

$$= \arg\min_i (R-m_k)^2$$

The following are the decision regions:

$$H_d = \begin{array}{ll} H_3 & -\frac{m}{2} < R < \frac{m}{2} \\ H_1 & R < -\frac{3m}{2} \\ H_2 & -\frac{3m}{2} < R < -\frac{m}{2} \\ H_4 & \frac{m}{2} < R < \frac{3m}{2} \\ H_5 & R > \frac{3m}{2} \end{array}$$

error probability $P(\varepsilon) = P(\varepsilon/H_3)\,P(H_3) + P(\varepsilon/H_2)\cdot P(H_2) + \cdots + P(\varepsilon/H_5)\cdot P(H_5)$

$$= \frac{1}{5}\left[Q\left(\frac{m}{2\sigma}\right) + 2\cdot Q\left(\frac{m}{2\sigma}\right) + 2\cdot Q\left(\frac{m}{2\sigma}\right) + 2\cdot Q\left(\frac{m}{2\sigma}\right) + Q\left(\frac{m}{2\sigma}\right)\right]$$

$$= \frac{8}{5}\, Q\left(\frac{m}{2\sigma}\right)$$