EE5130 - Assignment 2
Submissions due by 17/03

All the problems are from 'An Introduction to Signal Detection and Estimation' by Vincent Poor.

- **ch2,prob19** Consider the following pair of hypothesis concerning a sequence $Y_1, Y_2, \cdots, Y_n$ of independent Gaussian random variables

$$\mathcal{H}_0 \quad : \quad Y_k \sim \mathcal{N}\left(\mu_0, \sigma_0^2\right), \qquad k = 1, 2, \cdots, n$$

versus

$$\mathcal{H}_1 \quad : \quad Y_k \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right), \qquad k = 1, 2, \cdots, n$$

where μ_0, σ_0^2, μ_1 and σ_1^2 are known constants.

  (a) Show that likelihood ratio can be expressed as a function of the parameters μ_0, σ_0^2, μ_1 and σ_1^2 and the quantities $\sum_{k=1}^{n} Y_k^2$ and $\sum_{k=1}^{n} Y_k$.

  (b) Describe the Neyman-Pearson test for the two cases ($\mu_0 = \mu_1$ and $\sigma_1^2 > \sigma_0^2$) and ($\sigma_1^2 = \sigma_0^2$ and $\mu_1 > \mu_0$)

  (c) Find the thresholds and ROC's for the case $\mu_0 = \mu_1$ and $\sigma_1^2 > \sigma_0^2$ with $n = 1$.

- **ch2,prob20** Consider the hypothesis of above problem with $\mu \triangleq \mu_1 > \mu_0 = 0$ and $\sigma^2 \triangleq \sigma_0^2 > \sigma_1^2 > 0$. Does there exist a uniformly most powerful test of these hypothesis under the assumption that μ is known and σ^2 is not? If so, find it and show it is UMP. If not, show why and find the generalized likelihood ratio test.

- **ch3,prob7** Consider the hypothesis pair

$$\mathcal{H}_0 : Y_k = N_k, \; k = 1, 2, \cdots, n$$

versus

$$\mathcal{H}_1 : Y_k = N_k + \Theta S_k, \; k = 1, 2, \cdots, n$$

where $\underline{N} \sim \mathcal{N}\left(\underline{0}, \boldsymbol{C}\right), \underline{s}$ is known and Θ is a random variable independent of $\underline{N}$

  (a) Suppose that Θ is a discrete random variable taking $+1$ and -1 with equal probabilities(i.e, $\Pr(\Theta = +1) = \Pr(\Theta = -1) = 0.5$). Find optimal Neyman Pearson detector with probability of false alarm constraint $P_F \leq \alpha$ and find corresponding ROC. (`Hint:` $\cosh(x)$ is monotonically increasing in $|x|$.)

(b) Suppose that $\Theta \sim \mathcal{N}(0, \tilde{\sigma}^2)$. Assuming that noise covariance $\boldsymbol{C} = \sigma^2 \mathbf{I}$, show that likelihood ratio is of the form

$$\mathrm{L}\left(\underline{y}\right) = k_1 e^{k_2 \|\underline{s}^T \underline{y}\|^2}$$

where k_1 and k_2 are positive constants. Find k_2.

- **ch3,prob8** Suppose we have the observations, $Y_k = N_k + \theta S_k, k = 1, 2, \cdots, n$ where $N \sim \mathcal{N}(\underline{0}, \mathbf{I})$ and where $S_1, S_2, \cdots S_n$ are i.i.d. random variables independent of $\underline{N}$, and each taking values $+1$ and -1 with equal probabilities of 0.5.

  (a) Find the likelihood ration for testing $\mathcal{H}_0 : \theta = 0$ versus $\mathcal{H}_1 : \theta = A$, where A is a known constant.

  (b) For the case $n = 1$, find the Neyman Pearson rule and the corresponding detection probability for false alarm $\alpha \in (0, 1)$, for the hypothesis of (a).

  (c) Is there a UMP test of $\mathcal{H}_0 : \theta = 0$ versus $\mathcal{H}_1 : \theta \neq 0$ in this model? If so, why and what is it? If not, why not? Consider the cases $n = 1$ and $n > 1$ separately.

- **ch3,prob9** Consider an observed random n-dimensional vector $\underline{Y}$ that satisfies one of the two hypotheses,

$$\mathcal{H}_0 : \underline{y} = \underline{N}$$

versus

$$\mathcal{H}_1 : \underline{y} = \underline{N} + A\left[(1 - \Theta)\, \underline{s}^{(0)} + \Theta \underline{s}^{(1)}\right],$$

where $N \sim \mathcal{N}(\underline{0}, \mathbf{I})$; the quantity A is a positive, nonrandom scalar; the random parameter Θ is independent of $\underline{N}$ and takes the values 0 and 1 with equal probabilities of $\frac{1}{2}$; the signals $\underline{s}^{(0)}$ and $\underline{s}^{(1)}$ are known orthogonal signals, that is, they satisfy the condition

$$\sum_{k=1}^{n} s_k^{(l)} s_k^{(m)} = \begin{cases} 1 & \text{if}\, m = l \\ 0 & \text{if}\, m \neq l \end{cases}$$

  (a) Suppose the value of A is known. Find the likelihood ratio between hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$.

  (b) Consider a composite hypothesis testing problem:

$$\mathcal{H}_0 : A = 0$$

  versus

$$\mathcal{H}_1 : A > 0$$

  Show that a locally most powerful test of $\mathcal{H}_0$ versus $\mathcal{H}_1$ is given by

$$\sum_{k=1}^{n} y_k \left[s_k^{(0)} + s_k^{(1)}\right] \gtrless \eta$$

  (c) Find the receiver operating characteristics (P_D Vs P_F) of the detector given in part(b) for a given fixed value of A.

- **ch3,prob13** Consider the model

$$Y_k = \theta^{\frac{1}{2}} s_k R_k + N_k, k = 1, 2, \cdots, n$$

where $s_1, s_2, \cdots, s_n$ is a known signal sequence, $\theta \geq 0$ is a constant, and $R_1, R_2, \cdots, R_n$, $N_1, N_2, \cdots, N_n$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) Consider the hypothesis pair

$$\mathcal{H}_0 : \theta = 0$$

versus

$$\mathcal{H}_1 : \theta = A$$

where A is a known positive constant. Describe the structure of Neyman-Pearson detector.

(b) Consider now the hypothesis pair

$$\mathcal{H}_0 : \theta = 0$$

versus

$$\mathcal{H}_1 : \theta > 0$$

Under what conditions on $s_1, s_2, \cdots, s_n$, does a UMP exist?

(c) For the general sequence $s_1, s_2, ..., s_n$, find LMP.