Z-

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Z - Transform

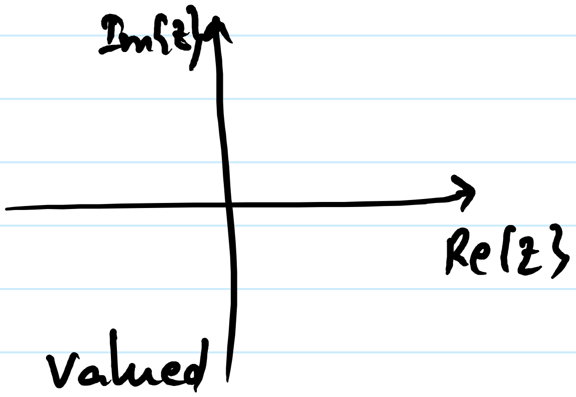
Analytical tool to study DT signals/ systems

$x(n)$ → DT signal

Z- transform of $x(n)$ denoted by $X(z)$ is defined as

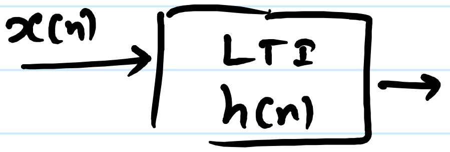
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

z → complex Variable
(takes values in complex plane)



$X(z)$ is a complex valued function of a complex variable

General Eigen functions of LTI systems



$y(n) = x(n) * h(n)$

Let $x(n) = z_0^n$, where

z_0 is a complex number

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, z_0^{n-k}$$

$$= z_0^n \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, z_0^{-k}}_{H(z_0)}$$

$H(z_0)$ | z transform of h(n) evaluated at $z = z_0$

$$y(n) = z_0^n\, H(z_0)$$

$\downarrow$

$z_0^n \rightarrow$ eigen function of LTI system.

Note: special case we saw

$$z_0 = e^{jw_0}$$

Notation $x(n) \xleftrightarrow{z} X(z)$

Examples:

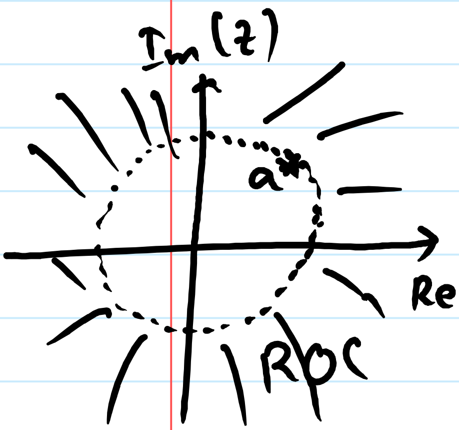
(causal sequence)
(right-sided)

① $x(n) = a^n u(n)$

where a is complex number

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$



$$= \frac{1}{1 - az^{-1}} \quad \text{if } |az^{-1}| < 1$$

$$X(z) = \frac{1}{1-az^{-1}}$$ with

validity for

$\underline{\underline{|z| > |a|}}$

Region of convergence (ROC) is the region in which z-transform is valid / defined properly.

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② $x(n) = -a^{n} u(-n-1)$

(left sided sequence)

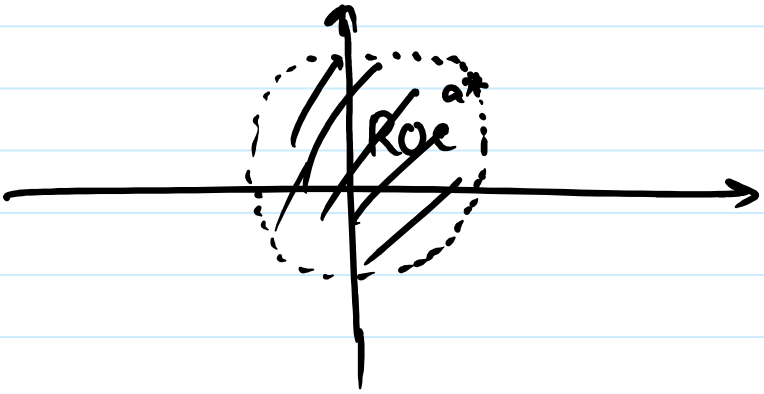
$$X(z) = \sum_{n=-\infty}^{-1} -a^{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} - a^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} + a^{-n} z^{n}$$

$$= 1 - \frac{1}{1-a^{-1}z} \quad \text{if } |a^{-1}z| < 1$$

$$X(z) = \frac{1}{1-az^{-1}} \quad \text{if } \underline{\underline{|z| < |a|}}$$



ROC is inside circle of radius |a|

Both examples 1 & 2 has same algebraic expression for X(z) but ROC is different.

Connection between z-transfrom & DTFT

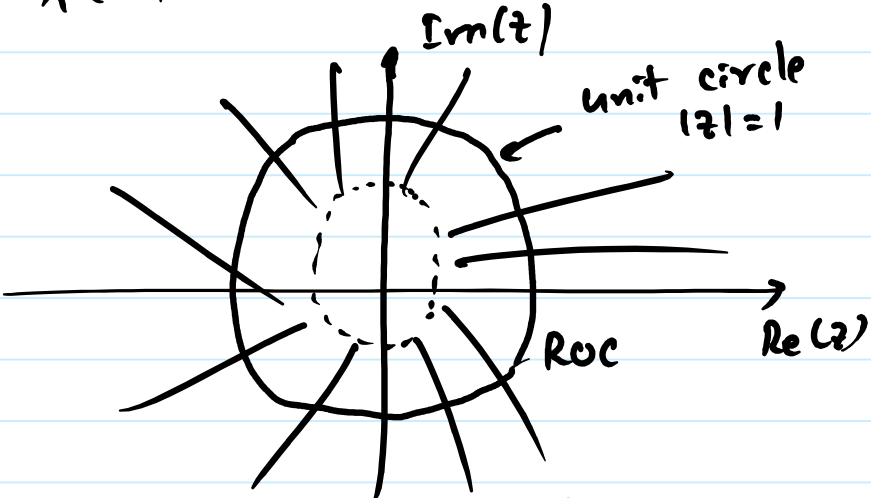
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

DTFT can be evaluated by Sampling z-transfrom along the unit circle

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} \quad \omega\in[-\pi,\pi]$$

DTFT of sequence $x(n)$ exists if and only if ROC of $X(z)$ includes unit circle.



For $a^n u(n)$ with $|a|<1$, we have DTFT exists.

Example ③

Finite duration sequence.

$$x(n) = \begin{cases} -1, & n=0 \\ 2, & n=1 \\ 5, & n=2 \\ \sqrt{2}, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

Notation

(a) $x(n) = \{\underset{\uparrow}{-1}, 2, 5, \sqrt{2}\}$

arrow indicates the sample corresponding to $n=0$

$$X(z) = -1\,z^{0} + 2z^{-1} + 5z^{-2} + \sqrt{2}\,z^{-3}$$

$$X(z) = -1 + 2z^{-1} + 5z^{-2} + \sqrt{2}\,z^{-3}$$

ROC is entire z-plane except the point $z=0$

(b) $x(n) = \{-1, 2, 5, \underset{\uparrow}{\sqrt{2}}\}$

$$X(z) = (-1)z^{3} + 2z^{2} + 5z + \sqrt{2}$$

ROC is entire z plane except $z=\infty$

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(c) $x(n) = \{-1, 2, \underset{\uparrow}{5}, \sqrt{2}\}$

$$X(z) = (-1) z^{2} + 2z + 5 + \sqrt{2} z^{-1}$$

ROC excludes both $z=0$ & $z=\infty$

Rational z-transform:

X(z) is called rational

$$\text{if} \quad X(z) = \frac{B(z)}{A(z)}$$

where $B(z)$, $A(z)$ are polynomials in z.

Roots of $A(z)$ are called <u>poles</u>

Roots of $B(z)$ are called <u>zeros</u>.

For instance

$$x(n) = a^{n} u(n)$$

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$= \frac{z}{z-a}$$

Pole $z=a$

Zero $z=0$

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Note: as $z \to a$, $X(z) \to \infty$

as $z \to 0$, $X(z) \to 0$

$\bullet \quad \longrightarrow$

Examples:

① $x(n) = \underbrace{\left(\frac{1}{2}\right)^n u(n)}_{x_1(n)} + \underbrace{\left(\frac{-1}{3}\right)^n u(n)}_{x_2(n)}$

$$X_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} , \quad |z| > 1/2$$

$$X_2(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} , \quad |z| > 1/3$$

$$Roc(x) = Roc(x_1) \cap Roc(x_2)$$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}$$

with $|z| > 1/2$

$$X(z) = \frac{2(1 - \frac{1}{12} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} , \quad |z| > 1/2$$

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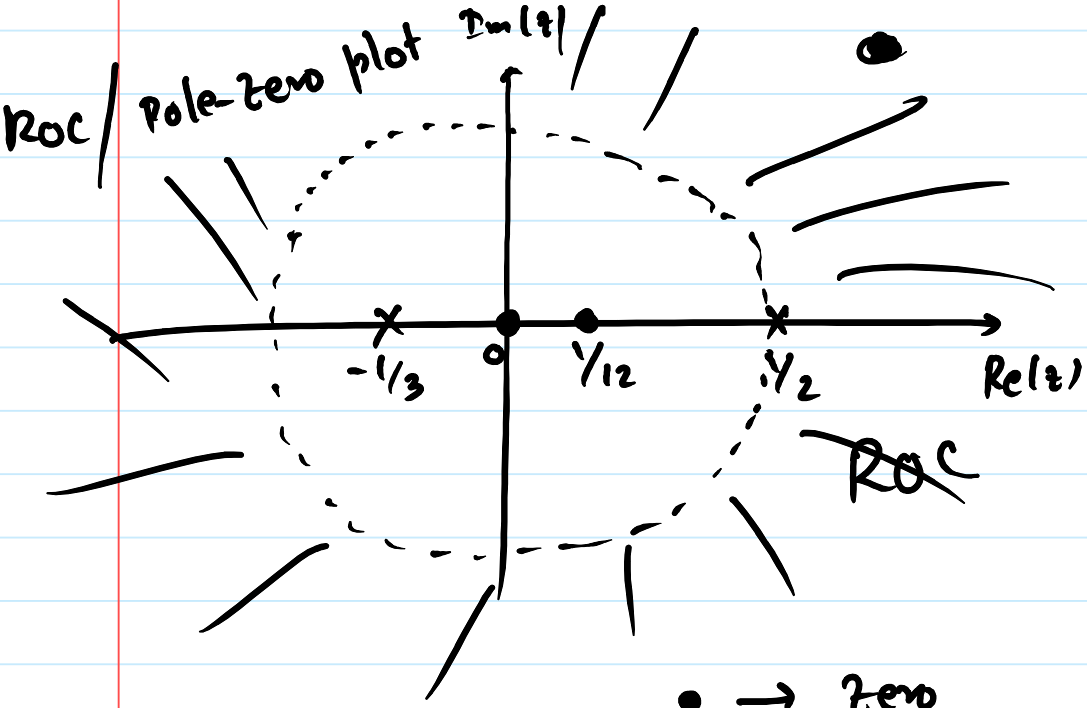
$$X(z) = \frac{2z\left(z - 1/12\right)}{\left(z - 1/2\right)\left(z + 1/3\right)}$$

Poles at $z = 1/2 , -1/3$

zeros at $z = 0, 1/12$


ROC / pole-zero plot

• → zero

x → pole

ROC is outside the pole with largest magnitude

($x(n)$ is right-sided)

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② $$x(n) = \overbrace{-\left(\tfrac{1}{2}\right)^n u(-n-1)}^{x_1(n)} \underbrace{-\left(-\tfrac{1}{3}\right)^n u(-n-1)}_{x_2(n)}$$

(left sided)

$$X_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} , \quad |z| < \tfrac{1}{2}$$

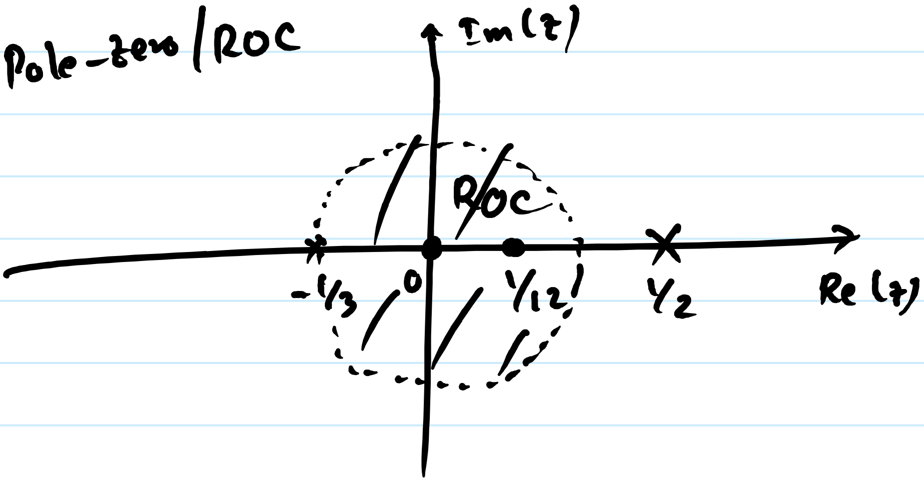
$$X_2(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} , \quad |z| < \tfrac{1}{3}$$

ROC of $X(z)$ is $|z| < \frac{1}{3}$

$\text{Roc}_{x_1} \cap \text{Roc}\ x_2$

$$X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}$$

Pole-zero/ROC



Roc lies inside the pole with smallest magnitude

(left-sided)

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Convergence of z-transform

$$x(n) \xleftrightarrow{z} X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$X(z)$ is called to converge absolutely if $|X(z)| < \infty$

Say $z = re^{j\omega}$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x(n)| r^{-n}$$

$$\leq \underbrace{\sum_{n=0}^{\infty} |x(n)| r^{-n}}_{\{\text{causal part}} + \underbrace{\sum_{n=-\infty}^{-1} |x(n)| r^{-n}}_{\text{anti-causal part}\}}$$

If $\exists r_0$ such that causal part converges then it also converges for any $r > r_0$

$n \geq 0$ $(r^{-n} < r_0^{-n})$ when $r > r_0$

If $\exists r_0$ such that anti-causal part converges, then it also converges for $r < r_0$

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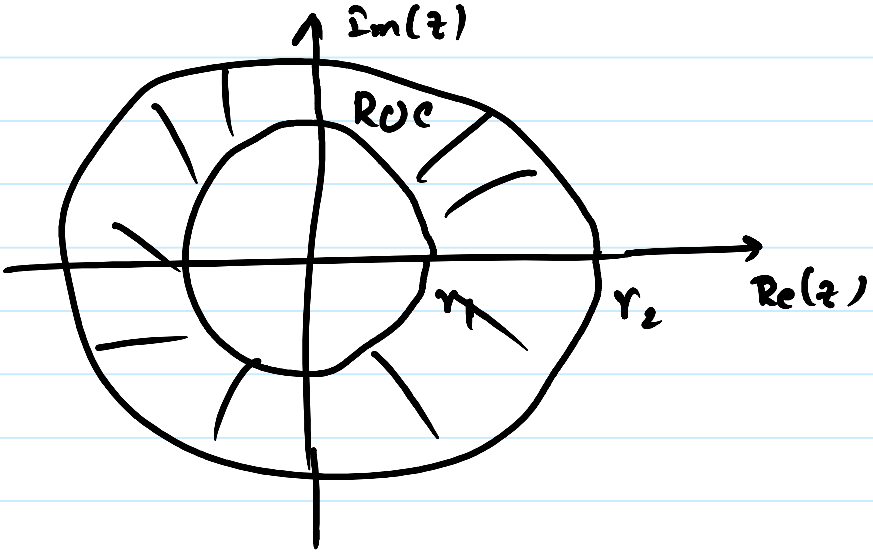
We want both causal & anticausal parts to converge

~~So, typically~~

So, typically ROC is of the form $r_1 < |z| < r_2$



r_1 can be as small as 0

r_2 can be as large as ∞

Remarks:

① only $|z|$ magnitude plays a role in convergence

$\angle z$ → angle does not affect convergence

If $r_0 \in$ ROC then $r_0 e^{j\omega} \in$ ROC for any ω.

② In general ROC is of form

$$r_1 < |z| < r_2$$

where inequality is strict

Note: If $x(n) = \frac{1}{n^2} u(n-1)$

then ROC inequality is not strict

(beyond our scope)

For rational z transforms the inequality is always strict.

Some more comments on Rational z-transforms

(a) For right sided sequence

$(x(n) = 0 \quad \forall n \le n_0)$

~~fixed num~~ arbitrary number

ROC is outside a circle.

1st. (b) For a left-sided sequence ROC lies <u>inside</u> a circle

(c) For a two-sided sequence ROC is an annular region

(d) ROC can not contain any poles

(e) ROC is always a connected region.

x ——— x

Remarks from (Complex Analysis)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$z \rightarrow$ complex variable

1. $X(z)$ is <u>analytic</u> in its ROC

means: $X(z)$ is differentiable and <u>all</u> higher order derivatives exist

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(in its ROC)

2. $X(z)$ satisfies Cauchy - Riemann equation

$z = x + jy$

$$X(z) = \underbrace{u(x+jy)}_{\text{real}} + j\underbrace{v(x+jy)}_{\text{imag.}}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{Cauchy Riemann equation}$$

Properties of z-transform

① Linearity

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

with ROC_{x_1}

~~$x_2(n)$~~

$$x_2(n) \longleftrightarrow X_2(z)$$

with ROC_{x_2}

$$x(n) = a x_1(n) + b x_2(n)$$

$$\xleftrightarrow{z} a X_1(z) + b X_2(z)$$

with

$$ROC_x \supseteq ROC_{x_1} \cap ROC_{x_2}$$

(contains)

ROC of x can be larger then $ROC_{x_1} \cap ROC_{x_2}$ (in some cases)

due to pole-zero cancellations.

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Example:

$x_1(n) = a^n u(n)$; $X_1(z) = \frac{1}{1-az^{-1}}$; $|z| > |a|$

$x_2(n) = a^n u(n-N)$; $X_2(z) = \frac{a^N z^{-N}}{1-az^{-1}}$; $|z| > |a|$

$x(n) = x_1(n) - x_2(n)$, $X(z) = X_1(z) - X_2(z)$

$= a^n [u(n) - u(n-N)]$ $\qquad = \frac{1}{1-az^{-1}}$

$\Downarrow$ $\qquad - \frac{a^N z^{-N}}{1-az^{-1}}$

finite duration sequence

$x(n) = 0$ if $n < 0$
or
$n \geq N$

$= \frac{1 - a^N z^{-N}}{1-az^{-1}}$

$\downarrow$

Pole at $z = a$ gets cancelled by zero at $z = a$

So ROC is entire z plane except origin ($z=0$)

Property 2 Time Shifting

$$x(n) \longleftrightarrow X(z)$$

$$x(n-N) \longleftrightarrow z^{-N} X(z)$$

Roc remains same except for possible to addition/deletion of $z=0$ or $z=\infty$

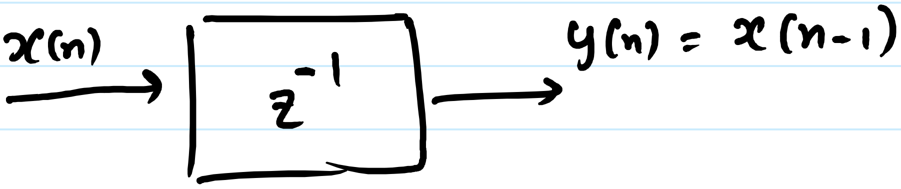
Recall

$x(n) \leftrightarrow X(e^{j\omega})$

$x(n-N) \leftrightarrow e^{-j\omega N} X(e^{j\omega})$

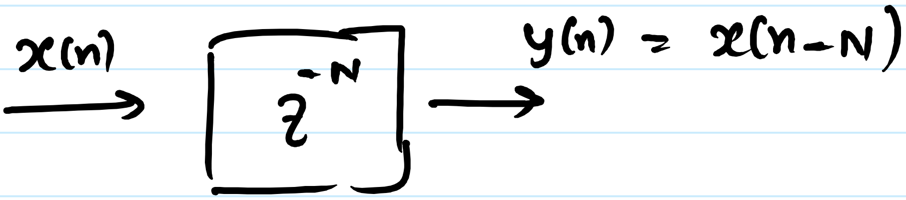
substitute $z = e^{j\omega}$ in the property

unit delay



⇓

z-domain representation of a delay system (unit)



⇓

delay of N units

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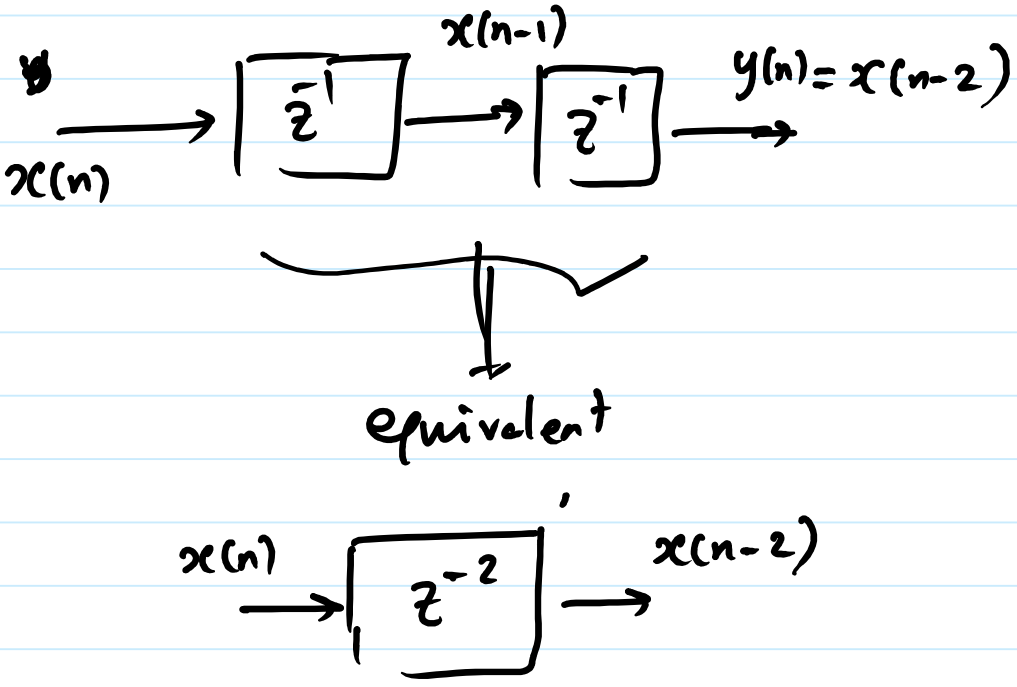
Suppose consider cascading of delays



Difference Equation & z-transform

Consider LCCDE System

$y(n) \rightarrow$ output

$x(n) \rightarrow$ input

$$a_0 y(n) + a_1 y(n-1) + \cdots + a_N y(n-N)$$

$$= b_0 x(n) + \cdots + b_M x(n-M)$$

(assume zero initial conditions)

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Taking z-transform

$$a_0 Y(z) + a_1 z^{-1} Y(z) + \dots + a_N z^{-N} Y(z)$$

$$= b_0 X(z) + \dots + b_M z^{-M} X(z)$$

Define $H(z) = \frac{Y(z)}{X(z)} = \frac{\text{output z-transform}}{\text{input z-transform}}$

$\downarrow$

Transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$\downarrow$

rational z-transform.

For a system characterized by LCCDE we have a rational transfer function

Similarly, for a system with rational transfer function, we have an equivalent LCCDE representation

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Property 3 Exponential Multiplication

$$x(n) \longleftrightarrow X(z) \; ; \; r_1 < |z| < r_2$$

Consider

$$y(n) = r^n x(n) \longleftrightarrow Y(z) = X(z/r)$$

r → complex number

$$|r| r_1 < |z| < |r| r_2$$

Proof:

$$Y(z) = \sum_n r^n x(n) z^{-n}$$

$$= \sum_n x(n) \left(z/r\right)^{-n}$$

$$= X(z/r)$$

↓

z-transform evaluated at z/r

z/r should lie in ROC of $X(z)$

$$r_1 < |z/r| < r_2$$

$$\Rightarrow |r| r_1 < |z| < |r| r_2$$

Recall Modulation property

$$\left.\begin{aligned} x(n) &\leftrightarrow X(e^{jw}) \\ e^{jw_0 n} x(n) &\leftrightarrow X(e^{j(w-w_0)}) \end{aligned}\right\} \begin{aligned} z &= e^{jw} \\ r &= e^{jw_0} \end{aligned}$$

What are the poles and zeros of $Y(z)$?

Suppose $X(z) = \frac{P(z)}{Q(z)}$

Suppose z_0 is a zero of $X(z)$

$$Y(z) = X(z/r)$$

$$= \frac{P(z/r)}{Q(z/r)}$$

$$Y(r z_0) = \frac{P(r z_0/r)}{Q(r z_0/r)}$$

$$= \frac{P(z_0)}{Q(z_0)}$$

$$= X(z_0)$$

$$= 0$$

Similarly if p_0 is a pole of $X(z)$ then $r p_0$ will be pole for $Y(z)$

Both poles & zeros get scaled by a factor of r.

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Property 4 Complex Conjugation

$$x(n) \longleftrightarrow X(z)$$

$$x^*(n) \longleftrightarrow X^*(z^*)$$

} Same ROC as $|z| = |z^*|$

z-transform of $x^*(n)$ at $z = \alpha$ is obtained as, $X^*(\alpha^*)$

↓

conjugate α & find z transform of $x(n)$ at $z = \alpha^*$ and then conjugate the z-transform output.

Recall

$$x(n) \longleftrightarrow X(e^{jw})$$

$$x^*(n) \longleftrightarrow X^*(e^{-jw})$$

Proof:

$$\sum_n x^*(n) z^{-n} = \left[\sum_n x(n) (z^*)^{-n} \right]^*$$

$$= X^*(z^*)$$

Symmetry in z-domain

Suppose $x(n)$ is real valued

$$x(n) = x^*(n)$$

So

$$X(z) = X^*(z^*)$$

Suppose z_0 is a zero of $X(z)$

$$\Rightarrow X(z_0) = 0$$

$$X^*(z_0^*) = 0$$

$$X(z_0^*) = 0$$

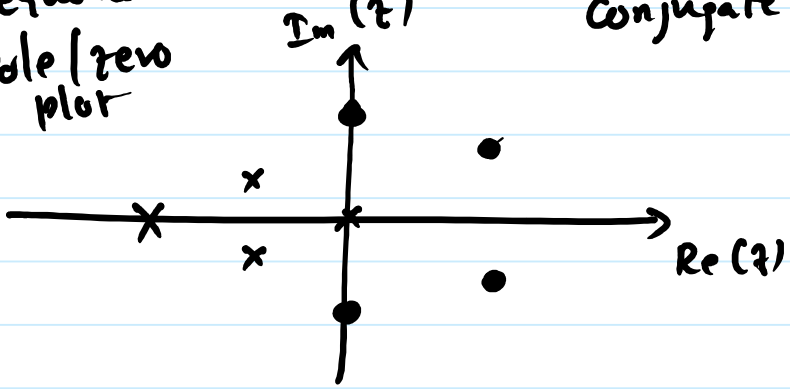
z_0^* is also a zero

Similarly if p_0 is a pole of $X(z)$

p_0^* will also be a pole

For real valued sequences, poles & zeros appear in complex conjugate pairs

Real sequence Pole/zero plot



Prop 5) Time reversal

$$x(n) \longleftrightarrow X(z) \qquad r_1 < |z| < r_2$$

$$x(-n) \longleftrightarrow X(z^{-1}) \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Proof.

$$\sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

$$\underset{l=-n}{=} \sum_{l=\infty}^{-\infty} x(l)\, z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\, (z^{-1})^{-l}$$

$$= X(z^{-1}) \qquad r_1 < |z^{-1}| < r_2$$

equivalently

$$\frac{1}{r_2} < |z| < \frac{1}{r_1}$$

x ——— x

If α is a pole (or zero) of $X(z)$ then $X(z^{-1})$ will have a pole (or zero) at $1/\alpha$

Symmetry

Suppose $x(n)$ is real valued and even signal

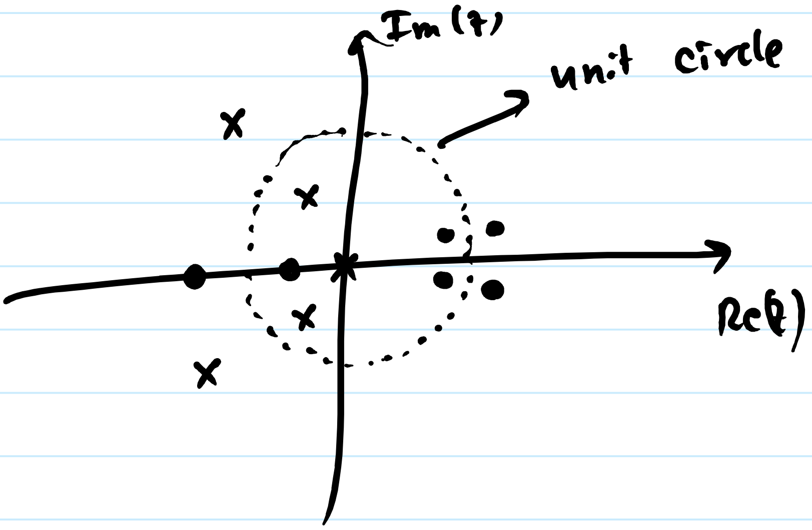
$$x(n) = x^*(n) = x(-n)$$

$$X(z) = X^*(z^*) = X(z^{-1})$$

So if α is a pole (or zero) of $X(z)$ then

α^*, $1/\alpha$, $\frac{1}{\alpha^*}$ are all poles (zeros)

Sample pole-zero plot for a real even signal



Prop. 6 Differentiation in z domain

$$x(n) \longleftrightarrow X(z)$$

$$n x(n) \longleftrightarrow -z \frac{dX(z)}{dz}$$

ROC is seme for rational transform

Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{d}{dz} X(z) = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n-1} (-n)$$

$$\Rightarrow z \frac{d}{dz} X(z) = \sum_{n} (-n) x(n) z^{-n}$$

x ——— x

Since X(z) is analytic, we can extend this to higher order derivatives also.

Example :

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} \; ; \quad |z| > |a|$$

$$n a^n u(n) \longleftrightarrow -z \frac{d}{dz} \frac{1}{1-az^{-1}} \; ; \quad |z| > |a|$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \; ; \quad |z|>|a|$$

Time shift by (+1)

$$(n+1) a^{n+1} u(n+1) \longleftrightarrow \frac{a}{(1-az^{-1})^2} \; ; \quad |z|>|a|$$

$$(n+1) a^n u(n) \longleftrightarrow \frac{1}{(1-az^{-1})^2} \; ; \quad |z|>|a|$$

x —— x

Exercise

$$? \longleftrightarrow \frac{1}{(1-az^{-1})^M} \; ; \quad |z|>|a|$$

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Prop 7 Convolution property

$$x(n) \longleftrightarrow X(z) \quad ROC_x$$

$$h(n) \longleftrightarrow H(z) \quad ROC_h$$

$$y(n) = x(n) * h(n) \longleftrightarrow$$

$$Y(z) = H(z)X(z)$$

$$ROC_y \supseteq ROC_x \cap ROC_h$$

Proof.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Y(z) = \sum_{n} y(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right] z^{-n} \frac{z^{k}}{z^{k}}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \underbrace{\sum_{n=-\infty}^{\infty} h(n-k)\, z^{-(n-k)}}_{H(z) \text{ for any } k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k}\, H(z)$$

$$= X(z)H(z) \qquad ROC_y \supseteq ROC_x \cap ROC_h$$

Example (a)

input $\rightarrow$ $x(n) = \left(\frac{1}{2}\right)^n u(n)$

$y(n) = x(n) * h(n)$

impulse response $\rightarrow$ $h(n) = \left(\frac{1}{3}\right)^n u(n)$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \; ; \; |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1 - (\frac{1}{3}) z^{-1}} \; ; \; |z| > \frac{1}{3}$$

$$Y(z) = \frac{1}{(1 - \frac{1}{3} z^{-1})} \frac{1}{(1 - \frac{1}{2} z^{-1})} \; ; \; |z| > \frac{1}{2}$$

$$= \frac{3}{1 - \frac{1}{2} z^{-1}} - \frac{2}{1 - \frac{1}{3} z^{-1}} \; ; \; |z| > \frac{1}{2}$$

$$y(n) = \underbrace{3 \left(\frac{1}{2}\right)^n}_{\text{input mode}} u(n) - \underbrace{2 \left(\frac{1}{3}\right)^n}_{\text{system mode}} u(n)$$

Example (b)

$$x(n) = (1/3)^n u(n)$$

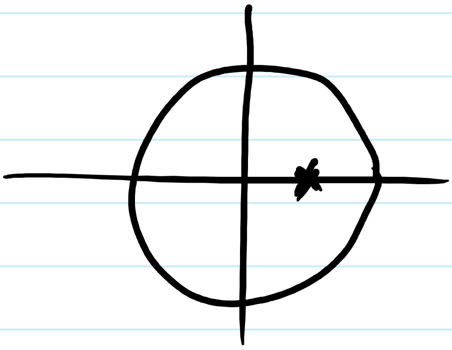
$$h(n) = (1/3)^n u(n)$$

input mode = system mode

(resonance !)

$$Y(z) = \frac{1}{(1-1/3 z^{-1})^2} \; ; \; |z| > \frac{1}{3}$$

(differentiation property)



?

$$y(n) = (n+1)\left(\frac{1}{3}\right)^n u(n)$$

$\downarrow$

high gain for large n

(due to resonance)

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Prop. 8. If $z=1$ is ROC of $X(z)$

Then

$$X(z)\Big|_{z=1} = X(1)$$

$$= \sum_{n=-\infty}^{\infty} x(n)$$

&

Equivalenty

DTFT $$X(e^{jw})\Big|_{w=0} = X(e^{j0})$$

$$= \sum_{n=-\infty}^{\infty} x(n)$$

x ———— x

Property 9 Initial Value Theorem

Suppose $x(n)$ is causal

$$x(n) = 0 \quad \forall n < 0$$

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$$

Taking $\lim_{z \to \infty} X(z) = x(0)$

Property 10 Final Value theorem

Suppose $x(n)$ is a causal sequence.

$$\lim_{z \to 1} X(z)(1 - z^{-1}) = \underset{\lim_{n \to \infty} x(n)}{\underset{\downarrow}{x(\infty)}}$$

Proof.

$$v(n) = x(n) - x(n-1)$$

$$V(z) = X(z) - z^{-1}X(z)$$

$$= (1 - z^{-1})X(z)$$

~~lim~~

$$V(z) = \sum_{n=-\infty}^{\infty} v(n) z^{-n}$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} v(n) z^{-n}$$

due to causality $\rightarrow$ $$= \lim_{N\to\infty} \sum_{n=0}^{N} v(n) z^{-n}$$

$$\lim_{z\to 1} V(z) = \lim_{z\to 1} \lim_{N\to\infty} \sum_{n=0}^{N} v(n) z^{-n}$$

$$= \lim_{N\to\infty} \lim_{z\to 1} \sum_{n=0}^{N} v(n) z^{-n}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} v(n)$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \left[x(n) - x(n-1)\right]$$

$$= \lim_{N\to\infty} \left[x(0) - x(-1)\right] + \left[x(1) - x(0)\right] + \dots + x(N-1) - x(N-2) + x(N) - x(N-1)$$

$$= \lim_{N\to\infty} x(N)$$

$$= x(\infty)$$

Example

(a) $x(n) = u(n)$

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$\lim_{z \to 1} (1-z^{-1}) X(z) = \lim_{z \to 1} \cdot \frac{1-z^{-1}}{1-z^{-1}}$$

$$= 1$$

$$= x(\infty)$$

(b) $x(n) = (-1)^n u(n)$

$$X(z) = \frac{1}{1+z^{-1}} ; \quad |z| > 1$$

$\lim_{n \to \infty} x(n)$ does not exist

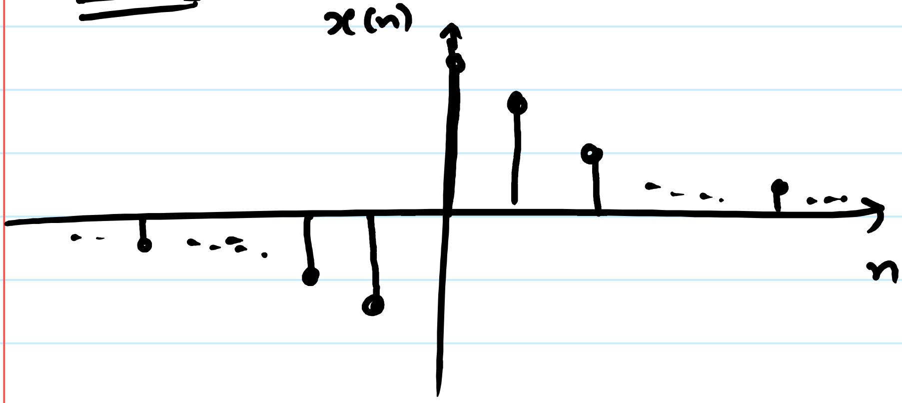
Final value theorem does not apply

DTFT of u(n)

$$x(n) = a^{n} u(n)$$

$$- a^{-n} (u-n-1)$$

$|a| < 1$



$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-a^{-1}z^{-1}}$$

$$|z| > |a|$$

We can find DTFT as

$$X(e^{jw}) = \frac{1}{1-ae^{-jw}} + \frac{1}{1-a^{-1}e^{-jw}}$$

$$\lim_{a \to 1} x(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ -1 & \text{for } n < 0 \end{cases}$$

$$= \text{Sgn}(n)$$

$$\Big\updownarrow \text{DTFT}$$

$$\frac{2}{1-e^{-jw}}$$

$$u(n) = \frac{1 + \text{sgn}(n)}{2}$$

$$U(e^{j\omega}) = \frac{1}{2}\left[\frac{2}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)\right]$$

$$= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

x ———————— x

Recall convolution property

$$\begin{tikzcd} y(n) \arrow[d, leftrightarrow, "z"] & = & x(n) \arrow[d, leftrightarrow, "z"] & * & h(n) \arrow[d, leftrightarrow, "z"] \\ Y(z) & = & X(z) & \cdot & H(z) \end{tikzcd}$$

$$\begin{tikzcd} h(n) \arrow[r, leftrightarrow, "z"] \arrow[d] & H(z) \arrow[d] \\ \text{Impulse response of LTI system} & \text{Transfer function} \end{tikzcd}$$

Causality

Suppose $h(n)$ is impulse response of LTI system

If system is causal,

$$h(n) = 0 \quad \forall n < 0$$

From initial value theorem

$$\lim_{z \to \infty} H(z) = h(0)$$

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots$$

$\lim_{z \to \infty} H(z)$ is finite

$\Rightarrow$ $z = \infty$ is in ROC

$\Rightarrow$ Since $h(n)$ is right sided ROC should be outside a circle.

Specifically for rational
transfer function

$$H(z) = \frac{P(z)}{Q(z)}$$

For causality:

① degree of $P(z) \leq$ degree of $Q(z)$

[$z = \infty$ is in ROC]

② ROC of $H(z)$ should be
outside the largest pole
(in magnitude)

* * *

Stability

LTI system with impulse response
$h(n)$ is stable

$\Rightarrow$ $h(n)$ is absolutely summable

$\Rightarrow$ DTFT of $h(n)$ exists

$\Rightarrow$ ROC of $H(z)$ should
include unit circle.

For a causal & stable system with rational transfer function

$$H(z) = \frac{P(z)}{Q(z)}$$

① Number of zeros (including multiplicity) $\leq$ Number of poles

② All roots of $Q(z)$ <u>must</u> lie inside unit circle

③ ROC should be outside largest pole.

Inverse Z-Transform

① Partial Fraction Method

→ Works for rational transforms

$$X(z) = \frac{P(z)}{Q(z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

degree of $P(z)$ = M

degree of $Q(z)$ = N

Suppose first r terms in Numerator

& first q terms in denominator are zero

the z^{-r} can be factored out in numerator

z^{-q} can be factored out in denominator

$$X(z) = \frac{z^{-r}}{z^{-q}} \frac{P_1(z)}{Q_1(z)}$$

$$= z^{-(r-q)} \frac{P_1(z)}{Q_1(z)}$$

WLOG assume

- $b_0 \neq 0$, $a_0 \neq 0$
- $a_0 = 1$

We have $X(z) = \dfrac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$

Case 1: $M < N$

(1a) Simple roots for $Q(z)$ (poles)

$$Q(z) = 1 + a_1 z^{-1} + \cdots + a_N z^{-N}$$

$$= \prod_{k=1}^{N} (1 - q_k z^{-1})$$

$q_1, q_2, \ldots, q_N$ are poles (they are distinct)

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We can write

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-q_k z^{-1}}$$

(residue)

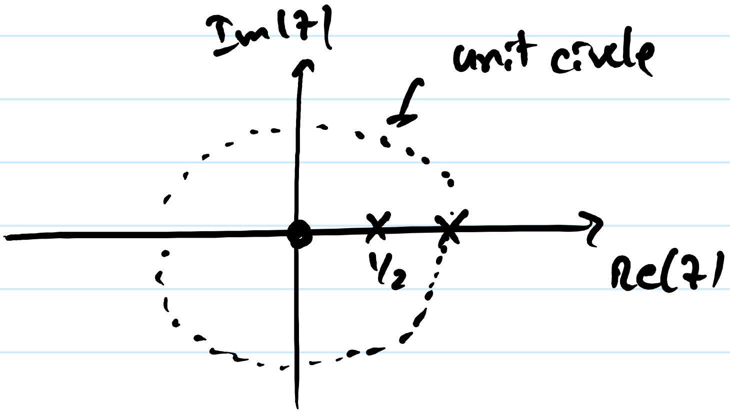
$$A_k = X(z)\left(1-q_k z^{-1}\right)\Big|_{z=q_k}$$

Example.

$$X(z) = \frac{1}{1-3/2\, z^{-1} + 1/2\, z^{-2}}$$

$$= \frac{1}{(1-z^{-1})(1-1/2\, z^{-1})}$$

$$= \frac{2}{1-z^{-1}} - \frac{1}{1-1/2\, z^{-1}}$$



To proceed further, we need ROC information

Case (i) $|z| < 1/2$

$x(n)$ is left sided

$$x(n) = -2\,u(-n-1) + (1/2)^n u(-n-1)$$

Case (ii) $1/2 < |z| < 1$

$x(n)$ is double sided

$$x(n) = -(1/2)^n u(n) - 2\,u(-n-1)$$

Case (iii) $|z| > 1$

$x(n)$ is right sided

$$x(n) = 2\,u(n) - (1/2)^n u(n)$$

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$M < N$

Case (1b) : Roots with multiplicity (poles)

$$X(z) = \frac{P(z)}{Q(z)} \quad Q(z) = 1 + a_1 z^{-1} + \dots + a_N z^{-N}$$

~~Set~~ Let $q_1, q_2, \dots, q_L$ are roots of $Q(z)$

Let $c_1, c_2, \dots, c_L$ are (order) multiplicity of corresponding root

Note $c_1 + c_2 + \dots + c_L = N$

So $$Q(z) = \prod_{k=1}^{L} \left(1 - q_k z^{-1}\right)^{c_k}$$

$X(z)$ can be written as

$$X(z) = \frac{P(z)}{Q(z)} \qquad \text{degree}(P) < \text{degree}(Q)$$

$$= \sum_{k=1}^{L} \sum_{m=1}^{c_k} \frac{A_{k,m}}{\left(1 - q_k z^{-1}\right)^{m}}$$

$$A_{k,m} = \frac{1}{(c_k - m)!} \frac{1}{(-q_k)^{c_k - m}} \cdot \left[\frac{d^{c_k - m}}{dw^{c_k - m}} \left((1 - q_k w)^{c_k} X(w^{-1}) \right) \right]$$

evaluated @ $w = q_k^{-1}$

Once we express $X(z)$ as sum of terms of form $\frac{A}{(1 - az^{-1})^m}$

Then we need ROC information to get exact sequence.

$m > 1$

$$\frac{(n+1)(n+2)\cdots(n+m-1)}{(m-1)!}\, a^n u(n) \longleftrightarrow \frac{1}{(1-az^{-1})^m} \; ; \quad |z| > |a|$$

(from repeated application of differentiation property)

$$? \longleftrightarrow \qquad \text{ROC } |z| < |a|$$

find yourself

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Case 2 $M \geq N$

$$X(z) = \frac{P(z)}{Q(z)}$$

$$\deg P(z) \geq \deg Q(z)$$

We can divide $P(z)$ by $Q(z)$
(long division)

~~Case~~

$$X(z) = \underbrace{\sum_{r=0}^{M-N} C_r z^{-r}}_{\text{quotient}} + \boxed{\frac{\text{Remainder}}{Q(z)}}$$

Remainder will have degree $< N$

(inverse)

Use previous method to invert this case

$$\sum_{r=0}^{M-N} C_r \delta(n-r)$$

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Example:

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-3/2\,z^{-1}+1/2\,z^{-2}}$$

$$= 2 + \frac{-1+5z^{-1}}{1-3/2\,z^{-1}+1/2\,z^{-2}}$$

$$= \underset{\substack{\downarrow \\ 2\delta(n)}}{2} + \underbrace{\frac{-9}{1-1/2\,z^{-1}} + \frac{8}{1-z^{-1}}}_{\Downarrow}$$

need ROC info

Three cases

(i) $|z| < 1/2$

(ii) $|z| > 1$

(iii) $1/2 < |z| < 1$

find $x(n)$ in each case.

07 March 2018 08:48

Power Series Method

(Inverse z- transform)

$$X(z) = \underbrace{\sum_{n=-\infty}^{\infty} x(n) z^{-n}}_{}$$

powers of z
& (z^{-1})

① $X(z) = z^{2}(1 - \frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$

$= z^{2} - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$

$= \{1, -\frac{1}{2}, \underset{\overset{\uparrow}{n=0}}{-1}, \frac{1}{2}\}$

② $X(z) = \frac{1}{1-az^{-1}}$; $|z| > |a|$

(right sided signal)

Power series →

$= 1 + az^{-1} + a^{2}z^{-2} + \cdots$

$= \{\ldots 0, 0, \underset{\uparrow}{1}, a, a^{2}, \cdots\}$

$= a^{n}u(n)$

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Long division

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad a^2z^{-2} \\
 & \qquad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad \cdots
\end{array}$$

Quotient in powers of z^{-1}

(right sided sequence)

Quotient gives the sequence.

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③ $X(z) = \dfrac{1}{1-az^{-1}}$; $|z| < |a|$

(left sided sequence)

$$= \frac{z}{z-a}$$

Write quotient as powers of z

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} \\
\hline
-a+z & z \\
 & z - a^{-1}z^{2} \\
\hline
 & +a^{-1}z^{2} \\
 & a^{-1}z^{2} - a^{-2}z^{3} \\
\hline
 & +a^{-2}z^{3} \\
 & --- \\
\hline
\end{array}
$$

$$X(z) = -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - \cdots$$

$$\underset{z}{\longleftrightarrow} \; -a^{n}u(-n-1)$$

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④

$$X(z) = \frac{1-a^2}{1+a^2 - a(z+z^{-1})} \; ;$$

$$|a| < 1$$

$$|a| < |z| < 1/|a|$$

$x(n) \to$ two sided

$X(z) \to$ break it into two portions

(partial fractions)

$$X(z) = \frac{1}{1-az^{-1}} + \frac{az}{1-az}$$

$\frac{1}{1-az^{-1}}$ $\Downarrow$ pole at a $\Downarrow$ right sided $\Downarrow$ long division in powers of z^{-1}

$\frac{az}{1-az}$ $\Downarrow$ pole at a^{-1} $\Downarrow$ left sided $\Downarrow$ long division in powers of z

$$= 1 + az^{-1} + a^2 z^{-2} + \dots \text{ (right sided)}$$

$$+ \; az\,(1 + az + a^2 z^2 + \dots) \text{ (left sided)}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \dots$$
$$+ \; az + a^2 z^2 + \dots$$

$z \to x(n) = a^{|n|}$

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⑤ $X(z) = e^{z} \quad ; \quad |z| < \infty$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

$$= \left\{ \ldots, \frac{2}{2!}, \frac{1}{1!}, \underset{\substack{\uparrow \\ n=0}}{1}, 0, 0, \ldots \right\}$$

Power series method works for non-rational transforms as well

Exercise

Find inverse of

$X(z) = \log(1 + a z^{-1})$

a) power series method

b) Differentiation property

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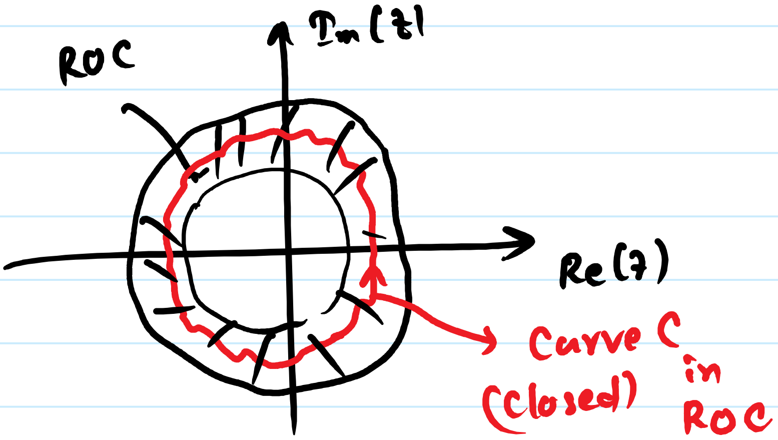
Contour Integration Method

(General form of
inverse z-transform)

Given X(z), the inversion formula

$$x(n) = \frac{1}{2\pi j} \oint_{C} X(z)\, z^{n-1}\, dz$$

↳ integration over a closed curve C in ROC



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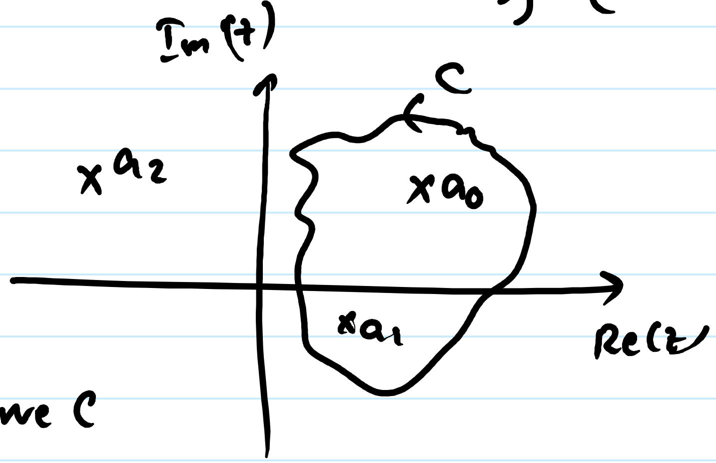
Computing the integral can be done using Cauchy's residue theorem.

~~If G(z)~~ f analytic function of z.

$$\frac{1}{2\pi j} \oint_{C} G(z)\, dz$$

= Sum of residues of G(z) at poles encircled by C

✓ example



a_0, a_1 are poles inside Curve C

$$\frac{1}{2\pi j} \oint_{C} G(z)\, dz = \text{residue at } a_0 + \text{residue at } a_1$$

Suppose G(z) has m^{th} order pole at a_0

G(z) can be written as

$$G(z) = \frac{\Gamma(z)}{(z-a_0)^m}$$

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residue at a_0

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Gamma(z) \Bigg|_{z=a_0}$$

x ———— x

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

= Sum of residues of $\left(X(z)\, z^{n-1} \right)$ at poles encircled by C.

x ———— x

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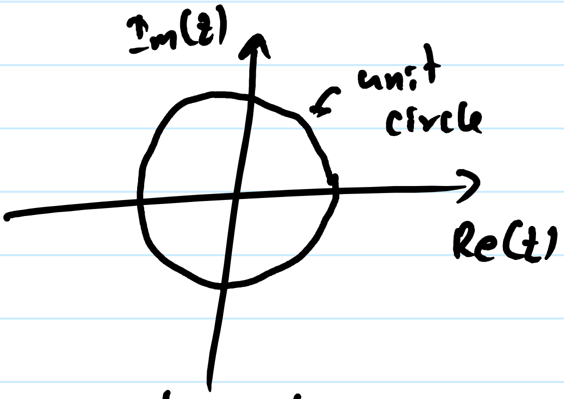
Inverse DTFT can be obtained using ~~[illegible]~~ Contour integral method.

$x(n)$ has DTFT $X(e^{jw})$

Then z-transfrom $X(z)$ will have ~~[illegible]~~ unit circle in the ROC



We can use unit circle as the curve C for contour integration

unit circle →

$$z = e^{jw} \quad ; \quad dz = \underbrace{je^{jw}}_{jz}\, dw$$

Change variables from z to w

~~dz~~

$$\frac{dz}{jz} = dw$$

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$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$= \frac{1}{2\pi} \oint_C X(z) z^{n} \frac{dz}{jz}$$

$$\boxed{x(n) = \frac{1}{2\pi} \int_{w=-\pi}^{\pi} X(e^{jw}) e^{jwn} dw}$$

inverse DTFT