Sam

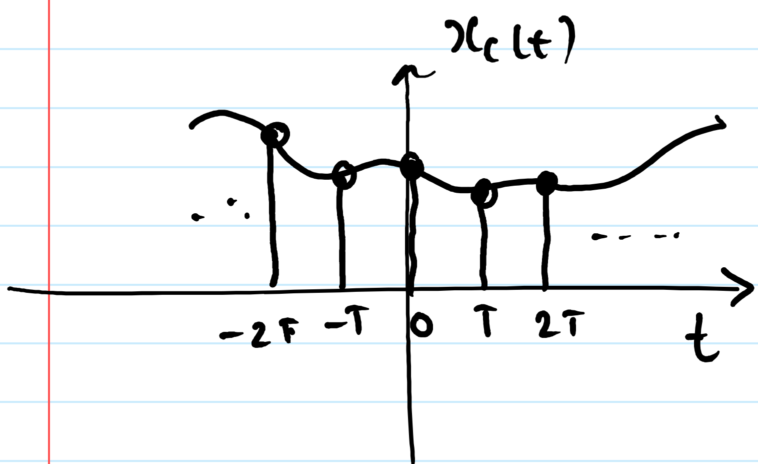
08 March 2018 15:43

Sampling

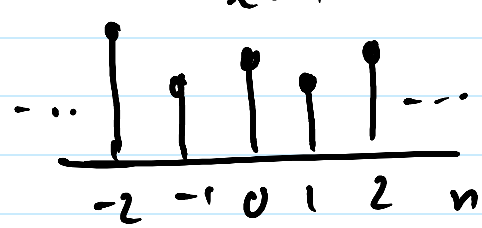
Theory behind converting :

Continuous-time signals to discrete-time signals and vice versa.

Consider continuous time signal $x_c(t)$



$x(n)$



Periodic Sampling:

Get samples of $x_c(t)$ periodically with interval T sec.

$$x(n) = x_c(nT), \quad n \in \mathbb{Z}$$

discrete time signal (containing samples of $x_c(t)$)

12 March 2018 11:06

Sampling Interval = T seconds

Sampling frequency = $\frac{1}{T}$ Hz = f_s

Sampling frequency (angular) = $\frac{2\pi}{T}$ radians/sec = $\underline{\underline{\Omega_s}}$

Spectrum of $x_c(t)$

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

Spectrum of $x(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Questions :

1. Can we <u>perfectly</u> reconstruct $x_c(t)$ from discrete time (sampled) signal $x(n)$?

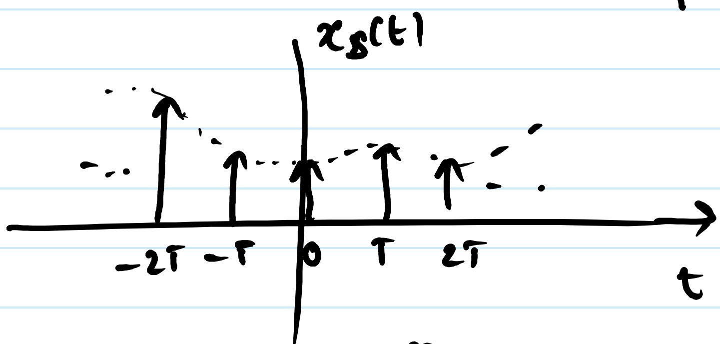
2. What is the relation between $X_c(j\Omega)$ & $X(e^{j\omega})$?

Consider an intermediate signal

$x_s(t)$ { has impulses at integer multiples of T

area under impulse at nT is $x_c(nT)$

$x_s(t)$ is 0 if t is not an integer multiple of T



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

Note: $x_s(t)$ is CT signal

$x(n)$ is DT signal

Both $x_s(t)$ & $x(n)$ have same information

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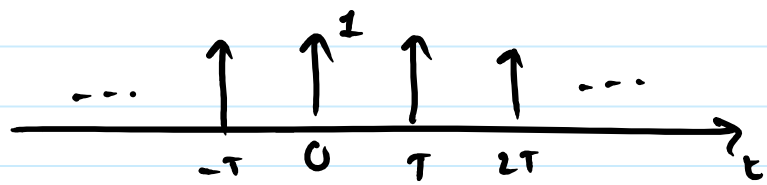
First, let us relate the spectrums $X_c(j\Omega)$ & $X_s(j\Omega)$

$x_s(t)$ is obtained by multiplying $x_c(t)$ with impulse train signal $s(t)$



$$s(t) = \sum_{n=-\infty}^{\infty} 1 \cdot \delta(t-nT)$$

$$x_s(t) = x_c(t) \cdot s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$x(t)\delta(t)$
$= x(0)\,\delta(t)$

$$= \sum_{n=-\infty}^{\infty} x_c(t)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

Now,

$$X_s(j\Omega) = \frac{1}{2\pi}\Big[\, X_c(j\Omega) \underbrace{*}_{\text{convolution}} S(j\Omega) \Big]$$

To find Spectrum $S(j\Omega)$

$s(t)$ is periodic signal

Fundamental Period T seconds

fundamental frequency $\Omega_s = \frac{2\pi}{T}$ seconds

We have Fourier series expansion

$$s(t) = \sum_{k=-\infty}^{\infty} \underbrace{s_k}_{\text{Fourier coefficient}} \underbrace{e^{jk\Omega_s t}}_{k^{th} \text{ harmonic}}$$

$$s_k = \frac{1}{T} \int_{-T/2}^{T/2} s(t)\, e^{-jk\Omega_s t}\, dt$$

$$= 1/T \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\Omega_s t}\, dt$$

$$= 1/T \; e^{-jk\Omega_s t} \Big|_{t=0}$$

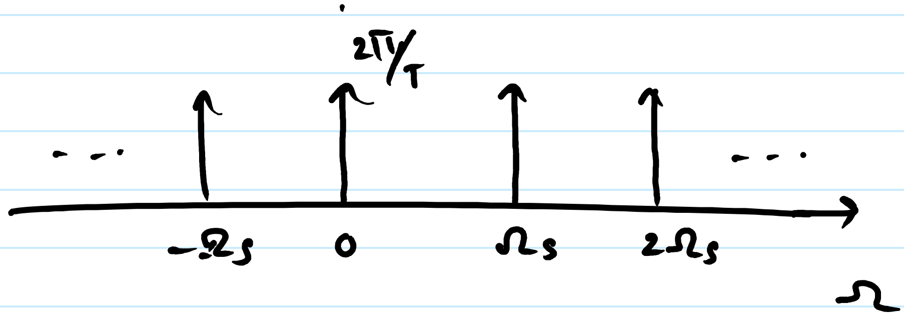
$$= 1/T \quad \text{for all } k.$$

$$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\Omega_s t}$$

$$e^{jk\Omega_s t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\Omega - k\Omega_s)$$

$$S(j\Omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi\,\delta(\Omega - k\Omega_s)$$

$S(j\Omega)$



$S(j\Omega) \Rightarrow$ impulse train in frequency domain

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$= \frac{1}{2\pi} X_c(j\Omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\Omega - k\Omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega) * \delta(\Omega - k\Omega_s)$$

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

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Remark :

$X_s(j\Omega)$ is periodic with

period Ω_s

(irrespective of $X_c(j\Omega)$)

$X_s(j(\Omega+\Omega_s))$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega+\Omega_s-k\Omega_s))$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega-(k-1)\Omega_s))$$

$$\overset{l=k-1}{=} \frac{1}{T}\sum_{l=-\infty}^{\infty} X_c(j(\Omega-l\Omega_s))$$

$$= X_s(j\Omega)$$

o ———————————— x

Under what conditions

~~$x_c(t)$~~ $x_s(t)$ has all

the information about $x_c(t)$?

13 March 2018 10:01

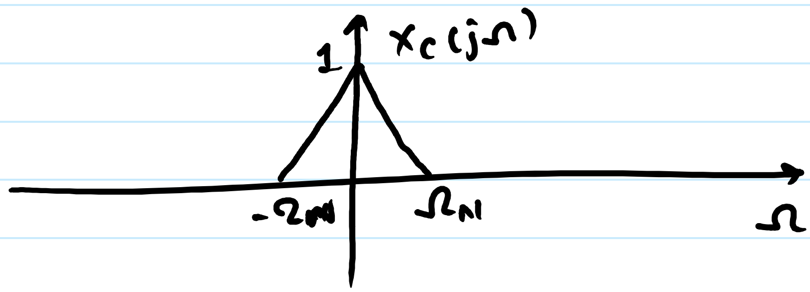
Consider the ~~c~~ band limited signal $x_c(t)$.

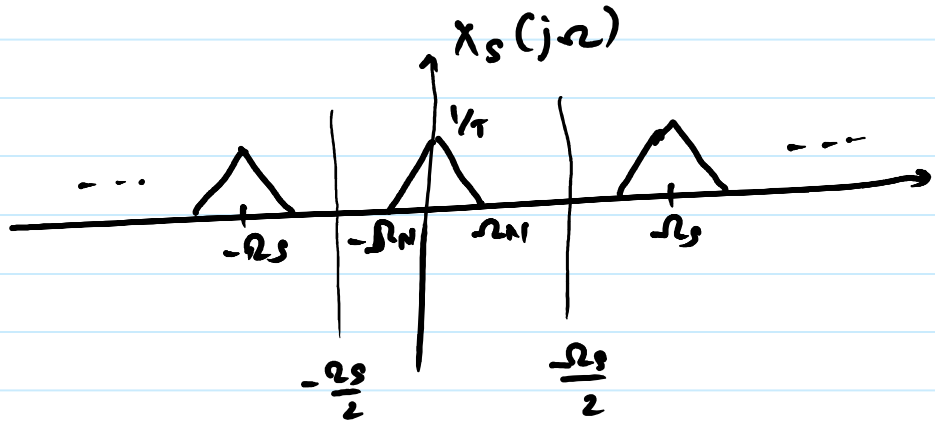
$$X_c(j\Omega) = 0 \text{ if } |\Omega| \geq \Omega_N$$

Ω_N → maximum frequency present in $x_c(t)$

Case ①

If sampling freq $\Omega_s \geq 2\Omega_N$





In interval $\Omega \in [-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}]$

spectrum $X_c(j\Omega)$ is identical to $X_s(j\Omega)$ except for scaling factor $1/T$

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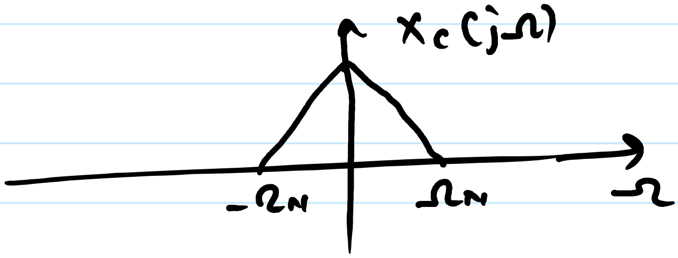
In this case

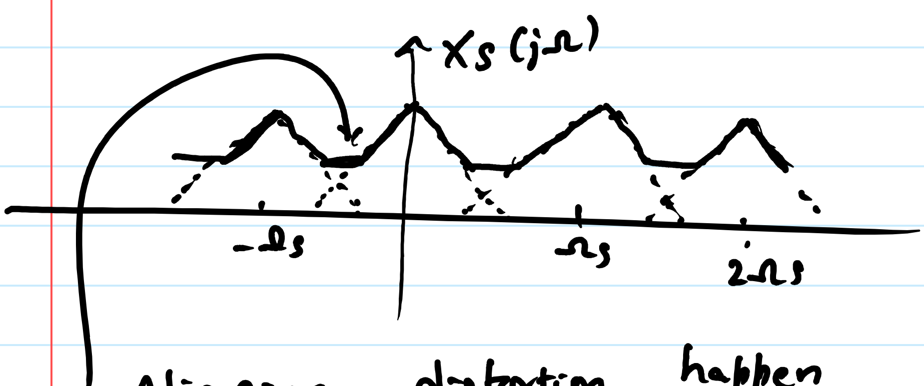
$x_s(t)$ & $x_c(t)$ have

same information

ie) we can get one from the other

<u>Case ②</u> $\Omega_s < 2\Omega_N$





Aliasing distortion happen

$x_s(t)$ do not have

all the info about $x_c(t)$

Sampling results in loss

of information.

13 March 2018 10:19

Nyquist - Shannon Sampling Theorem

Consider bandlimited signal $x_c(t)$

with $X_c(j\Omega) = 0$ if $|\Omega| \geq \Omega_N$

$x_c(t)$ is uniquely determined by its samples

$$x(n) = x_c(nT), \quad n \in \mathbb{Z}$$

as long as sampling freq

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

$\Omega_s = 2\Omega_N$ is called Nyquist sampling frequency

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Recall DT Signal

$$x(n) = x_c(t)\Big|_{@\, t = nT}$$

$$= x_c(nT), \quad n \in \mathbb{Z}$$

Note

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-nT)$$

Now

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} \quad — (*)$$

DT spectrum

CT spectrum

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_n x(n)\,\delta(t-nT)\, e^{-j\Omega t}\, dt$$

$$= \sum_n x(n) \int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\Omega t}\, dt$$

$$X_s(j\Omega) = \sum_n x(n)\, e^{-j\Omega T n} \quad — (**)$$

Comparing $(*)$, $(**)$, we have

$$X_s(j\Omega) = X(e^{jw})\Big|_{w = \Omega T}$$

Recall $X(e^{j\omega})$ is periodic with period 2π

$X_s(j\Omega)$ is periodic with period Ω_s

$$X_s(j\Omega) = X(e^{j\underbrace{\Omega T}_{\omega}})$$

$\Omega = 0 \qquad \Omega T = \omega = 0$

$\Omega = \Omega_s \qquad \Omega T = \omega = \Omega_s T$

$= \frac{2\pi}{T} \cdot T$

$= 2\pi$

We can write the DT spectrum $X(e^{j\omega})$ using original signal spectrum $X_c(j\Omega)$

$$X_s(j\Omega) = X(e^{j\Omega T})$$

$$= \frac{1}{T} \sum_{k=\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

Equivalently

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{k2\pi}{T}\right)\right)$$

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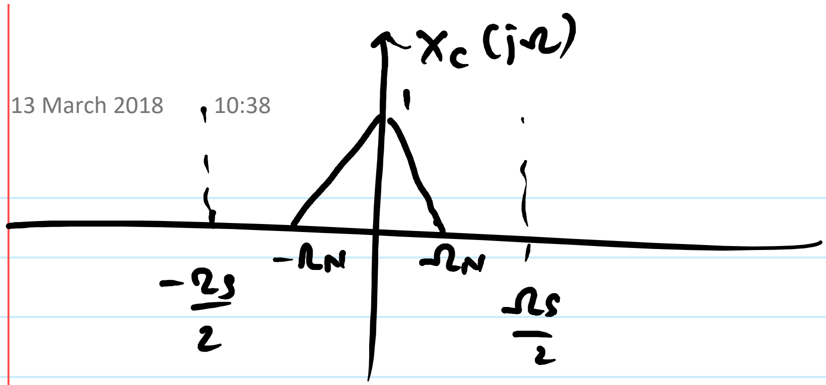
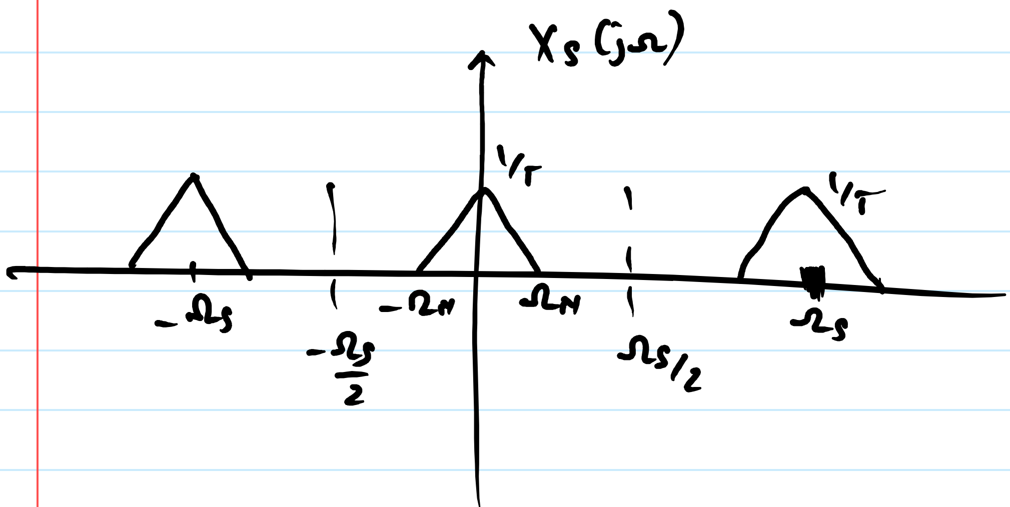
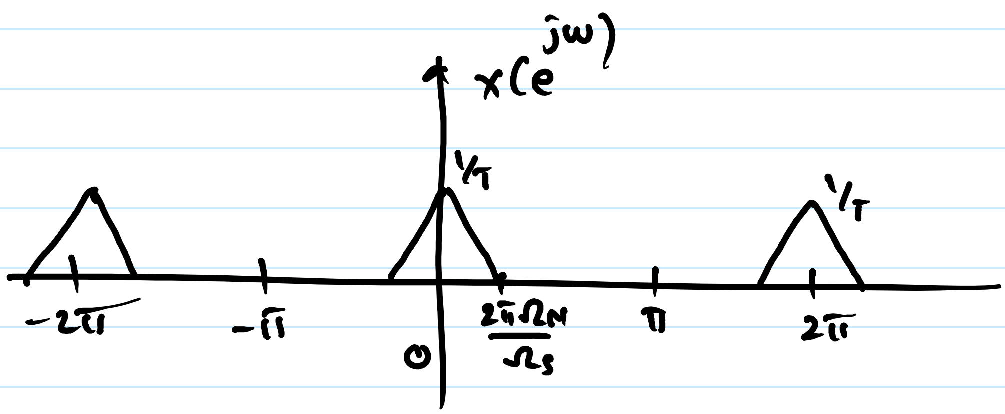






Reconstruction from Samples

$x_c(t) \rightarrow$ Continuous time Signal

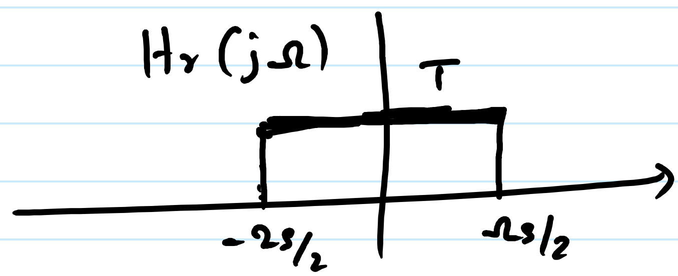
Sampling period = T sec.

$$x(n) = x_c(nT), \quad n \in \mathbb{Z}$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\, \delta(t-nT)$$

$\Downarrow$

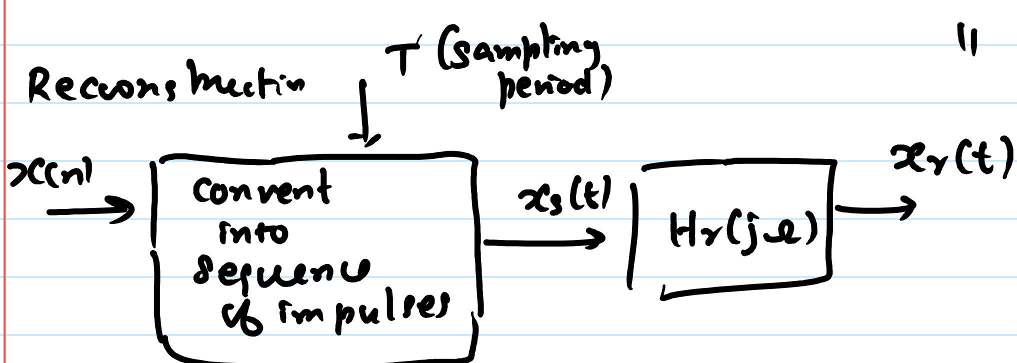
$x_s(t)$ is passed thru an ideal low pass filter with freq. response



$$\Omega s = 2\pi/T$$

$$\Omega s/2 = \pi/T$$

Reconstruction



$x_r(t) \; || \; x_c(t)$ if Nyquist Sampling criterion is met

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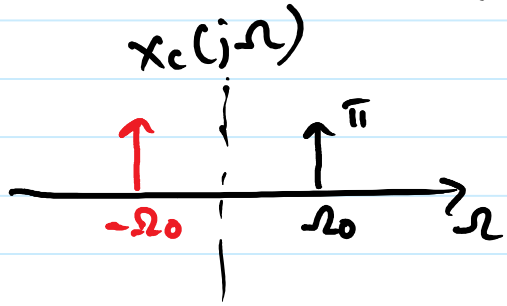
Example: Sampling of a Sinusoid

$$x_c(t) = \cos(\Omega_0 t)$$

$$= \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

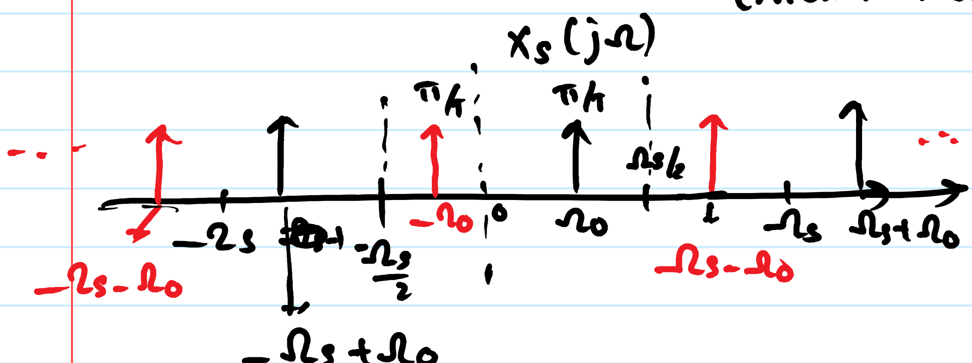
$$X_c(j\Omega) = \pi\,\delta(\Omega-\Omega_0) + \pi\,\delta(\Omega+\Omega_0)$$



$$x_s(t) = \sum_n x_c(nT)\,\delta(t-nT)$$

$$\Omega_s = 2\pi/T$$

If $\Omega_s > 2\Omega_0$ (Nyquist criterion met)



If $x_s(t)$ goes thro ideal reconstruction filter $H_r(j\Omega)$

we get back $x_c(t)$

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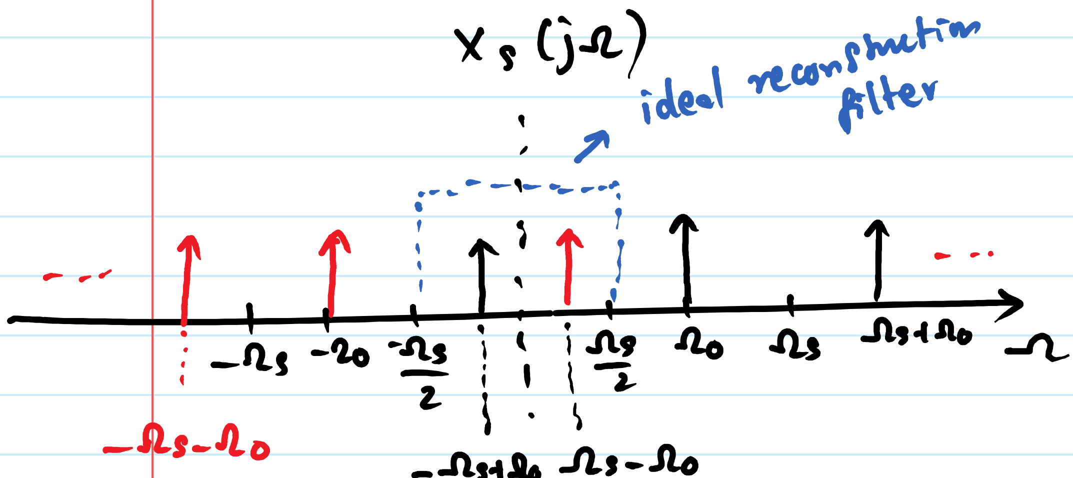
Case when

$$\frac{\Omega_s}{2} < \Omega_0 < \Omega_s$$

(Nyquist criterion not met)



Reconstructed signal

$$x_r(t) = \cos(\Omega_s - \Omega_0)t$$

Due to aliasing (Nyquist criterion not met)

$\cos(\Omega_0 t)$ after sampling

& reconstruction got

aliased as $\cos(\Omega_s - \Omega_0)t$

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Reconstruction of a CT Signal from its Samples

Suppose $x_c(t)$ is bandlimited from $-\Omega_N$ to Ω_N

$$X_c(j\Omega) = 0 \quad \text{if } |\Omega| > \Omega_N$$

Sampling period $= T$

Sampling freq. $\Omega_s = 2\pi/T$

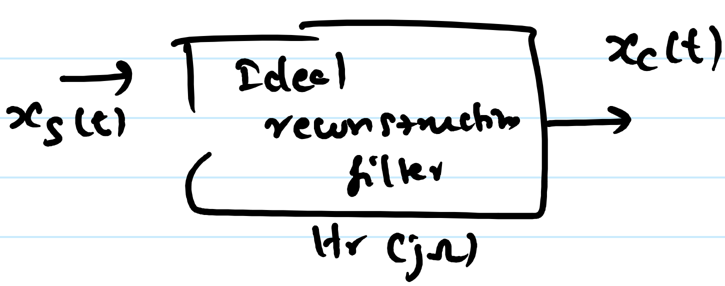
Let $\underline{\Omega_s > 2\Omega_N}$

Nyquist criterion

$$x_s(t) = \sum_n x_c(nT)\,\delta(t-nT)$$

$$x(n) = x_c(nT), \quad n \in \mathbb{Z}$$

Since Nyquist criterion met



$$H_r(j\Omega) = \begin{cases} T; & \text{if } |\Omega| < \Omega s/2 \\ 0 & \text{else} \end{cases}$$

$$\Omega s/2 = \pi/T$$

$h_r(t) \rightarrow$ impulse response

$$h_r(t) = \int_{-\infty}^{\infty} H_r(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \int_{-\pi/T}^{\pi/T} T\, e^{j\Omega t}\, d\Omega$$

$$h_r(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}$$

$$x_c(t) = x_s(t) * h_r(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t-nT)\right] * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, \underbrace{\delta(t-nT) * h_r(t)}_{h_r(t-nT)}$$

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Finally

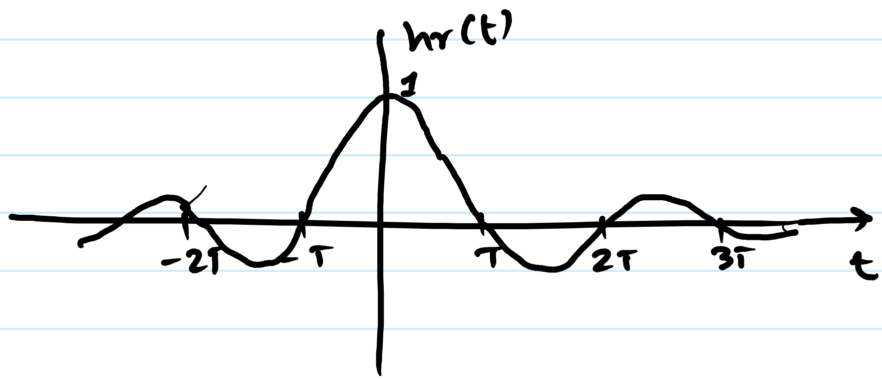
$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

reconstruction formula
(sinc interpolation)

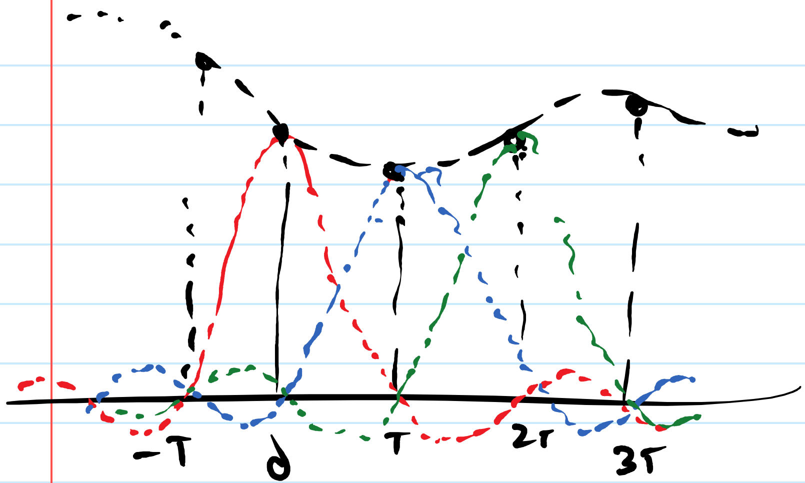
$$h_r(t) = \frac{\sin \pi t/T}{\pi t/T}$$

$h_r(0) = 1$ (L'Hospital's rule)

$h_r(nT) = 0$ for any interger $n \neq 0$



$$x_c(t) = \dots + x(-1) \frac{\sin(\pi(t+T)/T)}{\pi(t+T)/T} + x(0) \frac{\sin \pi t/T}{\pi t/T} + \dots$$



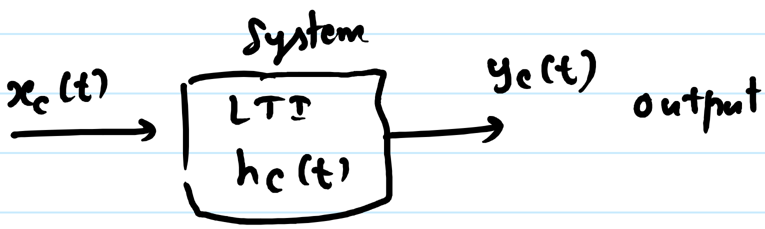
-T
0
T
2T
3T

Discrete-time processing (LTI) of CT Signals

$x_c(t)$ is bandlimited CT Signal

with $X_c(j\Omega) = 0$ if $|\Omega| \geq \Omega_N$

Say, $x_c(t)$ is given as an input to LTI System with impulse response $h_c(t)$ (frequency response $H_c(j\Omega)$

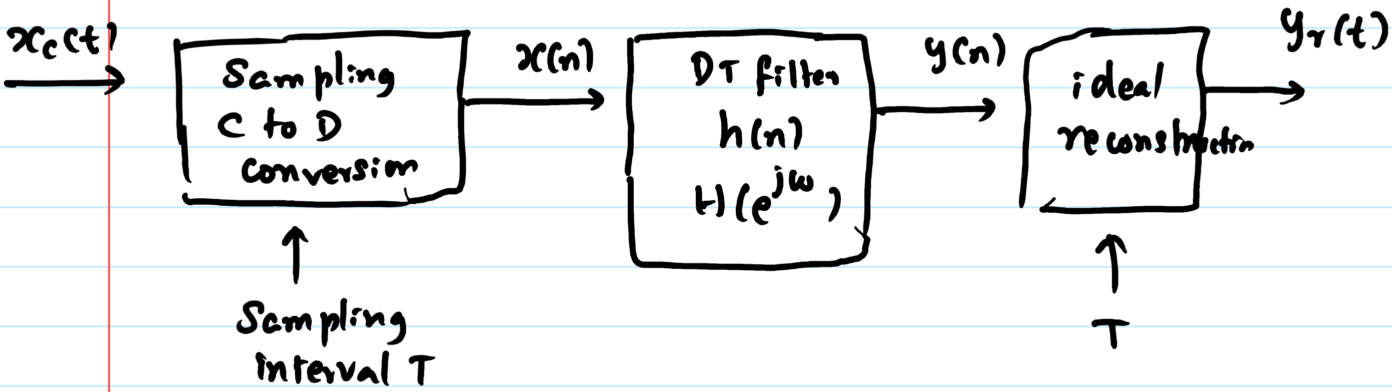


$$y_c(t) = h_c(t) * x_c(t)$$

$$Y_c(j\Omega) = H_c(j\Omega)\, X_c(j\Omega)$$

Can we build an equivalent system in Discrete time domain using Sampling & reconstruction?

Equivalent model



How to design DT system ($h(n)$, $H(e^{j\omega})$)

So that $y_r(t)$ coincides identically with $y_c(t)$?

- Sampling rate $\Omega_s = \frac{2\pi}{T}$

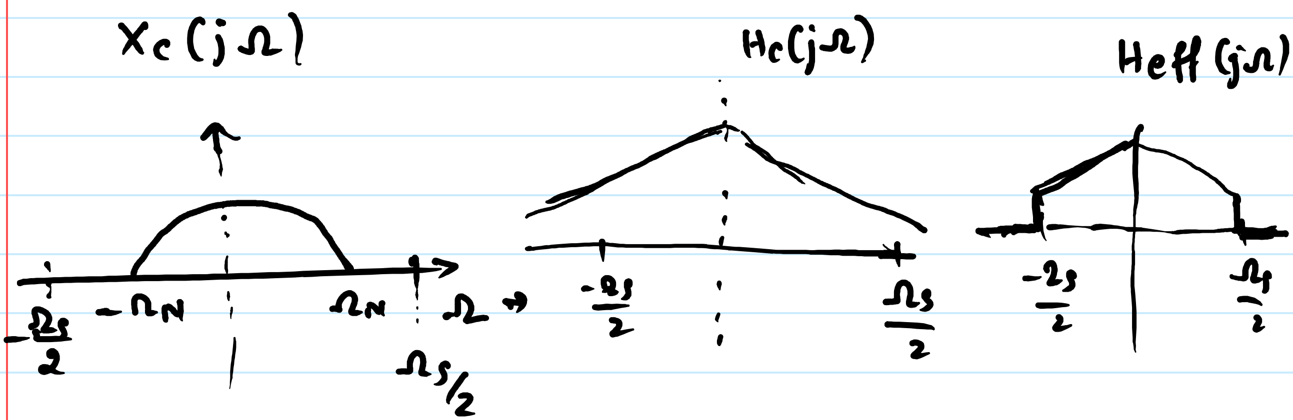
  Should be higher than

  Nyquist rate $2\Omega_N$

  (to avoid aliasing distortion)

19 March 2018 11:10

Various spectrums



$$Y_c(j\Omega) = H_c(j\Omega)\, X_c(j\Omega)$$

$$= \{ 0 \quad \text{if } |\Omega| \geq \Omega_N$$

Consider $H_{eff}(j\Omega) = \begin{cases} H_c(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\ 0 & \text{else} \end{cases}$

Note that since input is bandlimited

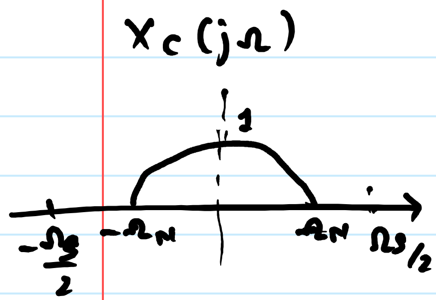
$$Y_c(j\Omega) = H_{eff}(j\Omega)\, X_c(j\Omega)$$

Want to build a DT filter $H(e^{j\omega})$ to match the effect of CT filter $H_{eff}(j\Omega)$

19 March 2018 11:19

CT Spectra

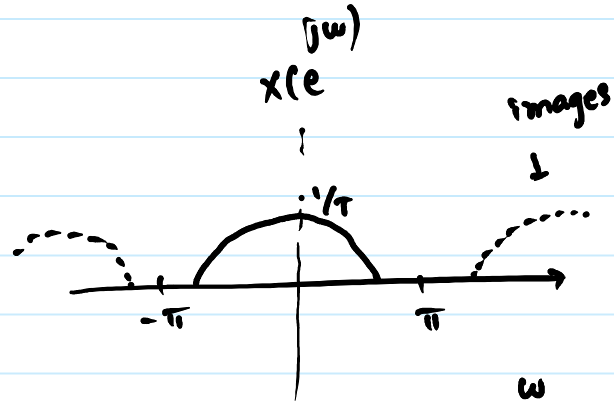
DT Spectra



Sampling

$$\Omega_s = \frac{2\pi}{T}$$

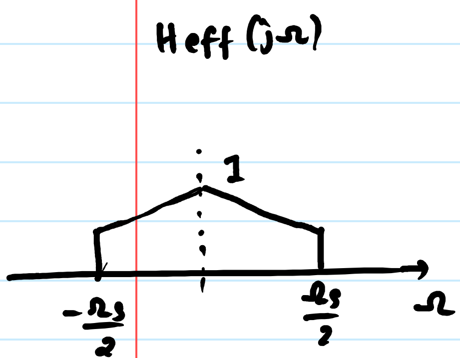
$$\Omega T = \omega$$



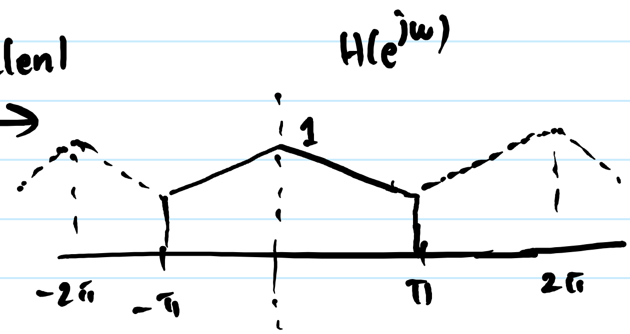
In interval $-\pi$ to π

$-\pi \le \omega \le \pi$

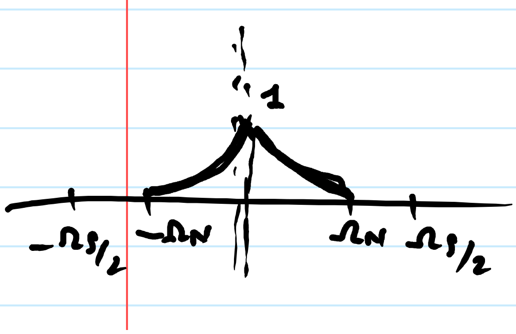
we have $X(e^{j\omega}) = \frac{1}{T} X_c(j\omega/T)$



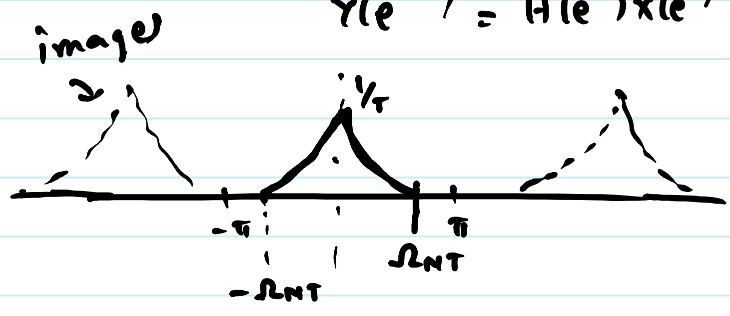
equivalent filter



$$Y_c(j\Omega) = X_c(j\Omega)\, H_{eff}(j\Omega)$$



$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$



Now

$$y(n) \rightarrow \boxed{\begin{array}{c} H_r(j\Omega) \\ \text{ideal} \\ \text{reconstruction} \\ \text{filter} \end{array}} \rightarrow y_r(t)$$

Note $\quad Y_r(j\Omega) = Y_c(j\Omega)$

Equivalences between DT system & CT system is $\quad \frac{\Omega_s}{2} = \frac{\pi}{T}$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) &, |\Omega| \le \pi/T \\ 0 &, \text{else} \end{cases}$$

Impulse invariance :

If $\quad H_c(j\Omega) = 0 \quad$ if $|\Omega| \geq \pi/T$

then the connection between CT & DT equivalent systems

$$H_c(j\omega/T) = H(e^{j\omega}) \; ; \; |\omega| \le \pi$$

$$T h_c(nT) = h(n)$$

21 March 2018 08:55

Impulse invariance applied for rational transfer functions

Consider $h_c(t) = e^{s_0 t} u(t)$

$Re\{s_0\} < 0$

CT freq. response

$$H_c(j\Omega) = \frac{1}{j\Omega - s_0}$$

Let us sample $h_c(t)$ with sampling period T

$$h(n) = T\, h_c(nT)$$

$$= T e^{s_0 n T} u(n)$$

$\rightarrow a^n u(n)$

$a = e^{s_0 T}$

DT freq. response

$$H(e^{j\omega}) = \frac{T}{1 - e^{s_0 T} e^{-j\omega}}$$

In this case, $H_c(j\Omega)$ is not bandlimited

Sampling introduces aliasing

Hence $H(e^{j\omega}) \neq H_c(j\omega/T)$

21 March 2018 09:05

CT equivalents for DT systems

Consider a DT system

with $h(n) = \delta(n-N)$

where N is integer

$$H(e^{j\omega}) = e^{-j\omega N} \quad ; |\omega| \le \pi$$

$$x(n) \longrightarrow \boxed{e^{-j\omega N}} \longrightarrow y(n) = x(n-N)$$

Consider DT system

$$H(e^{j\omega}) = e^{-j\omega\Delta} \quad ; |\omega| \le \pi$$

Δ is <u>NOT</u> integer

$$x(n) \longrightarrow \boxed{e^{-j\omega\Delta}} \longrightarrow y(n) \stackrel{?}{=} x(n-\Delta)$$

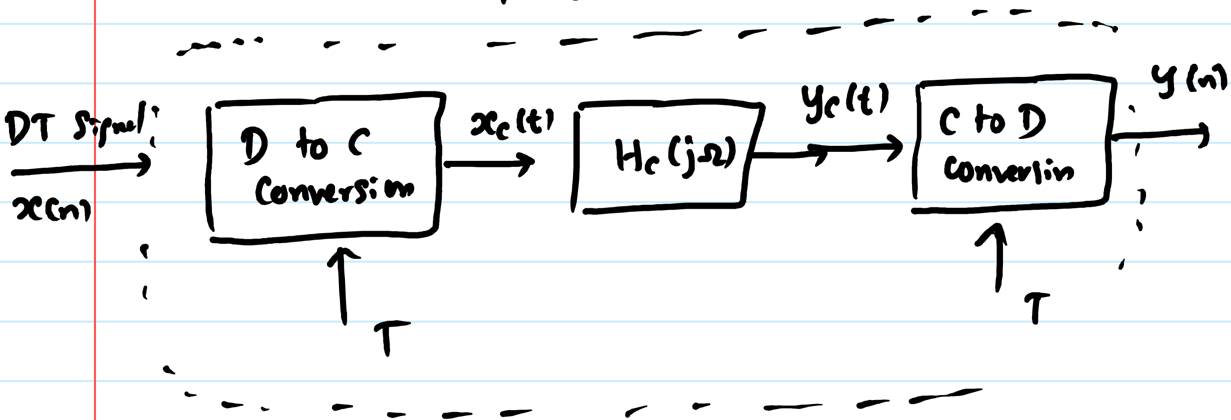
↓

no formal meaning if Δ is not integer

21 March 2018 09:11

Lets look at equivalent CT signals / systems

consider Sampling interval T



$|\omega| \leq \pi$; $H(e^{j\omega}) = H_c(j\omega/T)$

DT freq. response

Now $$x_c(t) = \sum_{k=-\infty}^{\infty} x(k) \frac{\sin[\pi(t-kT)/T]}{\pi(t-kT)/T}$$

$$y_c(t) = \sum_{k} y(k) \frac{\sin(\pi(t-kT)/T)}{\pi(t-T)/T}$$

Due to ideal reconstruction (LPF with cutoff $-\pi/T$ to π/T)

$x_c(t)$ & $y_c(t)$ will be bandlimited from $-\pi/T$ to π/T

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Without loss of generality

we can take

$H_c(j\Omega) = 0$ if $|\Omega| \geq \pi/T$

Relating DT & CT spectrums (No aliasing)

$$\frac{1}{T} X_c(j\Omega) = X(e^{j\Omega T}) ;$$

$$|\Omega| \leq \pi/T$$

$$Y_c(j\Omega) = X_c(j\Omega) H_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c(j\omega/T) \; ; \; |\omega| \leq \pi$$

$$= \frac{1}{T} X_c(j\omega/T) H_c(j\omega/T)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H_c(j\omega/T)$$

Equivalent DT system response is

$$|\omega| \leq \pi ; H(e^{j\omega}) = H_c(j\omega/T)$$

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Consider $H(e^{j\omega}) = e^{-j\omega\Delta}$

$\Downarrow$

equivalent CT system

$$H_c(j\Omega) = H(e^{j\Omega T})$$

$$= e^{-j\Omega T\Delta}$$

$$x_c(t) \longrightarrow \boxed{e^{-j\Omega T\Delta}} \longrightarrow y_c(t) = x_c(t - T\Delta)$$

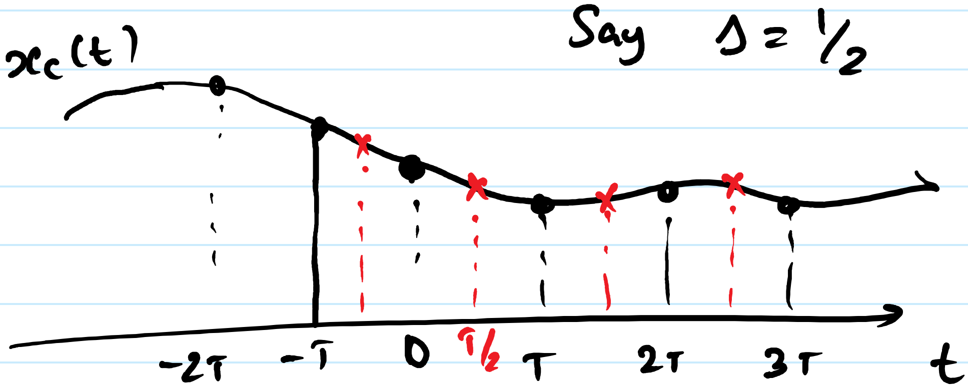
DT System output

$$y(n) = y_c(nT)$$

$$= x_c(t - T\Delta)\Big|_{t = nT}$$

$$= x_c(nT - T\Delta)$$

Say $\Delta = 1/2$



$\bullet \rightarrow x(n)$ $\quad$ $* \rightarrow y(n)$